ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **6:4** (2026) Article #S2R33
https://doi.org/10.54550/ECA2026V6S4R33

Multidimensional Continued Fractions and Riordan Arrays

Rachel Perrier

Franciscan University of Steubenville, 1235 University Blvd., Steubenville, OH 43952, USA
rperrier@franciscan.edu

Received: February 6, 2025, Accepted: December 19, 2025, Published: July 17, 2026

ABSTRACT: Continued fractions allow us to represent real numbers in terms of nested fractions, and these fractions have been analyzed in many different contexts, including connections to Riordan arrays. However, since the 1800s, mathematicians have expanded the concept of continued fractions to multidimensional continued fractions, which use matrices to encode algorithms for these analogs of continued fractions. We examine a connection between purely periodic continued fractions produced by the Jacobi–Perron algorithm and the combinatorial construction of Riordan arrays. We then expand this connection to relate multidimensional continued fractions to families of Riordan arrays in the case of purely periodic multidimensional continued fractions of period 1.

Keywords: Jacobi-Perron algorithm; Multidimensional continued fractions; Riordan arrays
2020 Mathematics Subject Classification: primary 05A15; secondary 11A55

1 Introduction

A Riordan array $A = \mathcal{R}(d(t), h(t))$ has entries

$$a_{n,k} = [t^n]\big(d(t)h(t)^k\big),$$

where $d(t)$ and $h(t)$ are power series satisfying $d(0) \neq 0$, $h(0) = 0$, and $h'(0) \neq 0$. Here, $[t^n]f(t) = f_n$ is the extraction operator that gives the coefficient of t^n in the series expansion of $f(t)$. The Riordan property guarantees that A is lower triangular; that is, for integers n and k with $0 \leq n < k$, we have $a_{n,k} = 0$.

Riordan arrays were first considered as a generalization of Pascal's triangle [20, 25, 26]. The Pascal-triangle Riordan array is

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Riordan arrays have many combinatorial properties, but for the scope of this paper we focus our attention on their row sums. Since Riordan arrays are lower triangular, each row sum consists of finitely many nonzero entries. The closed form for the row sums in terms of $d(t)$ and $h(t)$ was introduced in [27]:

$$\sum_{k=0}^{\infty} d_{n,k} = [t^n]\frac{d(t)}{1-h(t)}. \tag{1}$$

We have altered equation (1) to reflect our slight difference in the definition of $h(t)$ from that used in [27].

Our goal is to explore connections between Riordan arrays and multidimensional continued fractions. Although continued fractions were not originally studied from a combinatorial angle, Flajolet examined them from this perspective [11]. More recent research has analyzed connections between continued fractions and Riordan arrays [3–5, 12]. In Section 2, we give a brief overview of continued fractions and multidimensional continued fractions. We then discuss matrix analogs of these fractions in Section 3. In Sections 4 and 5, we use matrix analogs of periodic multidimensional continued fractions to connect these fractions to families of Riordan arrays determined by their row sums. This work was developed as part of the author's doctoral dissertation [22], and some text and results may resemble or reproduce portions of the original dissertation.

2 Overview of Continued Fractions and Multidimensional Continued Fractions

Although there are many types of continued fractions, we focus our attention on ordinary continued fractions. An ordinary continued fraction is a real number of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where $a_i \in \mathbb{Z}$ for all i, and a_i is positive for every $i > 0$ [17]. The notation

$$[a_0; a_1 : a_2 : \cdots : a_n]$$

is used to represent an ordinary continued fraction. This definition has finitely many a_i values and therefore describes ordinary continued-fraction representations of rational numbers. We can also form representations

$$[a_0; a_1 : a_2 : \cdots]$$

with an infinite sequence of a_i values to describe irrational numbers. In these representations, fractions are continually added in the denominators. Our focus in this paper is on infinite ordinary continued fractions, although some initial results apply to both the finite and infinite cases.

All numerators are 1 in an ordinary continued fraction. Other types of continued fractions may have numerators other than 1 or may use subtraction rather than addition in their definitions [10, 11, 15, 29]. Unless stated otherwise, "continued fraction" means "ordinary continued fraction" throughout this paper.

The following algorithm, adapted from [17], finds the continued-fraction representation of a real number.

Input. Start with $\alpha \in \mathbb{R}$.
Step 1. If $\alpha \in \mathbb{Z}$, then $\alpha = [\alpha]$. If $\alpha \notin \mathbb{Z}$, write

$$\alpha = \lfloor \alpha \rfloor + \frac{1}{1/(\alpha - \lfloor \alpha \rfloor)}.$$

Let $a_0 = \lfloor \alpha \rfloor$ and $r_1 = 1/(\alpha - \lfloor \alpha \rfloor)$.
Inductive Step k. Suppose the process has been performed $k-1$ times and has produced values through a_{k-2} and r_{k-1}. Then

$$r_{k-1} = \lfloor r_{k-1} \rfloor + \frac{1}{1/(r_{k-1} - \lfloor r_{k-1} \rfloor)}.$$

Thus, $a_{k-1} = \lfloor r_{k-1} \rfloor$ and $r_k = 1/(r_{k-1} - \lfloor r_{k-1} \rfloor)$.
Output. Stop once r_n is an integer. The output is

$$\alpha = [a_0; a_1 : \cdots : a_{n-1} : a_n].$$

If r_n is never an integer, as occurs when α is irrational, the algorithm does not terminate.

Example 1. Suppose we want to find the continued-fraction representation of 19/4. The algorithm gives

$$\frac{19}{4} = 4 + \frac{1}{4/3} = 4 + \cfrac{1}{1 + \cfrac{1}{3}},$$

so

$$\frac{19}{4} = [4; 1 : 3].$$

We are interested in continued fractions because they provide alternative representations of real numbers, as expressed by the following theorem. We focus on representations of positive numbers, although negative numbers also have continued-fraction representations.

Theorem 1. *Every real number $r > 0$ can be written uniquely as a continued fraction*

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where a_0 is a nonnegative integer and all remaining a_i are positive integers.

A proof of the corresponding result for all real numbers appears as Theorem 1.17 in [17].

Since the algorithm for an irrational number never terminates, one may ask how its continued-fraction representation helps us analyze the number. The k-convergents provide one answer. They consist of a pair of polynomials P_k and Q_k representing the first k iterations [17]:

$$\frac{P_k(x_0,\ldots,x_k)}{Q_k(x_0,\ldots,x_k)} = [x_0; x_1 : \cdots : x_k], \qquad P_k(0,\ldots,0) + Q_k(0,\ldots,0) = 1.$$

Let $p_k = P_k(a_0,\ldots,a_k)$ and $q_k = Q_k(a_0,\ldots,a_k)$ for an infinite continued fraction $[a_0; a_1 : \cdots]$ or a finite continued fraction $[a_0; a_1 : \cdots : a_n]$. The following recursive relationship will be used in Section 3.

Proposition 1 (Proposition 1.13 of [17])**.** *For every integer k,*

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2}, \\ q_k = a_k q_{k-1} + q_{k-2}. \end{cases}$$

The k-convergents of the continued-fraction representation of a real number α provide the best Diophantine approximations to α [17]. Even when the full representation of an irrational number cannot be computed in finitely many steps, convergents still provide best approximations.

Continued-fraction representations of irrational numbers have also been studied to determine which produce periodic patterns. A continued fraction is periodic if there are positive integers k_0 and h such that $a_{k+h} = a_k$ for every $k > k_0$ [17]. For example,

$$\sqrt{2} = [1; \overline{2}]$$

is periodic, whereas the well-known expansions

$$e = [2; 1 : 2 : 1 : 1 : 4 : 1 : 1 : 6 : \cdots] \quad \text{and} \quad \pi = [3; 7 : 15 : 1 : 292 : 1 : \cdots]$$

are not periodic. We indicate the periodic portion with an overline.

Lagrange proved that eventual periodicity occurs if and only if the continued fraction represents a quadratic irrational, that is, a number of the form

$$\frac{a + b\sqrt{c}}{d},$$

where $a, b, c, d \in \mathbb{Z}$, $b \neq 0$, $d > 0$, $c > 1$, and c is square-free [18].

In 1839, Hermite asked whether structures with similar periodic patterns could be found for cubic irrationals [9]. This question initiated the study of higher-dimensional analogs of continued fractions. Multidimensional continued fractions work with a pair of numbers, or more generally an n-tuple, depending on the dimension under consideration. A complete description of all pairs or n-tuples of real numbers that produce eventually periodic multidimensional continued fractions is not known.

Many algorithms generalize the process of finding continued-fraction representations. One of the best known is the Jacobi–Perron algorithm. We state the version used in [18].

Input. Start with $(x_0, y_0, z_0) \in \mathbb{R}^3$.
Step. If the previous step produced (x_i, y_i, z_i), define

$$(x_{i+1}, y_{i+1}, z_{i+1}) = \left(y_i, z_i - \left\lfloor \frac{z_i}{y_i} \right\rfloor y_i, x_i - \left\lfloor \frac{x_i}{y_i} \right\rfloor y_i \right).$$

The ith element of the multidimensional continued fraction is

$$\left(\left\lfloor \frac{z_i}{y_i} \right\rfloor, \left\lfloor \frac{x_i}{y_i} \right\rfloor \right).$$

Termination. Stop when $y_i = 0$.

This is an extension of the algorithm for ordinary continued fractions. To study the interaction of two numbers, one takes $x_0 = 1$. The Jacobi–Perron algorithm can also be extended to more than three inputs, producing higher-dimensional continued fractions. For more information about the Jacobi–Perron algorithm, including periodicity results and applications, see [2, 6, 16, 23]. For other multidimensional continued-fraction algorithms, see [24].

3 Matrix Representations

The steps of the Jacobi–Perron algorithm may be viewed as a sequence of vectors in $\mathbb{R}^3$. More generally, continued fractions and multidimensional continued fractions can be represented by sequences of vectors and products of matrices. In this section, we first represent ordinary continued fractions using matrices and then extend the construction to multidimensional continued fractions.

3.1 Matrix Representations of Continued Fractions

We consider continued-fraction representations of positive numbers, although the ideas can be extended to negative numbers. Consider matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}, \qquad k \in \mathbb{Z}_{>0}.$$

Products of these matrices have the form

$$\prod_{j=0}^{n} \begin{bmatrix} 0 & 1 \\ 1 & k_j \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where $k_j \in \mathbb{Z}_{>0}$ for $0 \le j \le n$ and $\alpha, \beta, \gamma, \delta$ are integers. Since each factor has determinant -1,

$$\alpha\delta - \beta\gamma = (-1)^{n+1}.$$

Another property of the convergents, found in the proof of Proposition 1.15 in [17], is

$$p_{k-1}q_k - p_k q_{k-1} = (-1)^k,$$

which motivates the following proposition.

Proposition 2. *For every integer $n \ge 1$,*

$$\prod_{j=0}^{n} \begin{bmatrix} 0 & 1 \\ 1 & k_j \end{bmatrix} = \begin{bmatrix} Q_{n-1}(k_0, \ldots, k_{n-1}) & Q_n(k_0, \ldots, k_n) \\ P_{n-1}(k_0, \ldots, k_{n-1}) & P_n(k_0, \ldots, k_n) \end{bmatrix},$$

where $k_j \in \mathbb{Z}_{>0}$.

Proof. We proceed by induction. For $n = 1$,

$$\begin{bmatrix} 0 & 1 \\ 1 & k_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & k_1 \end{bmatrix} = \begin{bmatrix} 1 & k_1 \\ k_0 & 1 + k_0 k_1 \end{bmatrix} = \begin{bmatrix} Q_0(k_0) & Q_1(k_0, k_1) \\ P_0(k_0) & P_1(k_0, k_1) \end{bmatrix}.$$

Assume the result holds for some $n \ge 1$. Then

$$\begin{aligned} \prod_{j=0}^{n+1} \begin{bmatrix} 0 & 1 \\ 1 & k_j \end{bmatrix} &= \begin{bmatrix} Q_{n-1} & Q_n \\ P_{n-1} & P_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & k_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} Q_n & Q_{n-1} + k_{n+1} Q_n \\ P_n & P_{n-1} + k_{n+1} P_n \end{bmatrix} \\ &= \begin{bmatrix} Q_n(k_0, \ldots, k_n) & Q_{n+1}(k_0, \ldots, k_{n+1}) \\ P_n(k_0, \ldots, k_n) & P_{n+1}(k_0, \ldots, k_{n+1}) \end{bmatrix}, \end{aligned}$$

where the final step follows from Proposition 1. □

Thus, multiplication of these 2×2 matrices encodes the convergents of $[k_0; k_1 : \cdots]$, or of $[k_0; k_1 : \cdots : k_n]$ in the finite case. These matrices provide a matrix analog of continued fractions.

We can also use eigenvectors and eigenvalues to recover Lagrange's result. Let

$$\begin{bmatrix} 1 \\ x \end{bmatrix}$$

be an eigenvector of

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with eigenvalue λ. Then

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \alpha + \beta x \\ \gamma + \delta x \end{bmatrix},$$

so

$$\lambda = \alpha + \beta x = \frac{\gamma + \delta x}{x}.$$

Therefore,

$$\beta x^2 + (\alpha - \delta)x - \gamma = 0,$$

and the roots are

$$\begin{aligned} x &= \frac{\delta - \alpha \pm \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{2\beta} \\ &= \frac{\delta - \alpha \pm \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2\beta} \\ &= \frac{\delta - \alpha \pm \sqrt{(\alpha+\delta)^2 - 4(-1)^{n+1}}}{2\beta}. \end{aligned}$$

Let

$$\xi = \frac{\delta - \alpha + \sqrt{(\alpha+\delta)^2 - 4(-1)^{n+1}}}{2\beta}$$

be the positive root. If

$$\left(\prod_{j=0}^{n} \begin{bmatrix} 0 & 1 \\ 1 & k_j \end{bmatrix}\right)^m = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^m = \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix},$$

then

$$\frac{\delta_m}{\beta_m} = [k_0; k_1 : \cdots : k_n : k_0 : k_1 : \cdots : k_n : \cdots : k_0 : k_1 : \cdots : k_n],$$

with m repetitions of the block $k_0, k_1, \ldots, k_n$. Moreover,

$$\lim_{m\to\infty} \frac{\delta_m}{\beta_m} = \xi,$$

and hence

$$\xi = [\overline{k_0; k_1 : \cdots : k_n}].$$

Thus, the matrix representation preserves Lagrange's result that periodic continued fractions represent quadratic irrationals.

3.2 Matrix Representations for Multidimensional Continued Fractions

We now use a 3×3 matrix to encode an equivalent version of the Jacobi–Perron algorithm, and then indicate how the construction generalizes to $k \times k$ matrices. The notation in this subsection largely follows [13].

Let $\alpha, \beta \in \mathbb{R}_{\geq 0}$, $a = \lfloor \alpha \rfloor$, and $b = \lfloor \beta \rfloor$. Consider

$$M = \begin{bmatrix} -a & 1 & 0 \\ -b & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}.$$

Then

$$M^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

is the companion matrix of

$$p(t) = t^3 - bt^2 - at - 1,$$

and

$$Mv = \begin{bmatrix} \alpha - a \\ \beta - b \\ 1 \end{bmatrix}.$$

Our goal is to construct a sequence of vectors whose first entry is always 1. If $\alpha \notin \mathbb{Z}$, then

$$\frac{1}{\alpha - a} Mv = \begin{bmatrix} 1 \\ (\beta - b)/(\alpha - a) \\ 1/(\alpha - a) \end{bmatrix}.$$

If $\alpha \in \mathbb{Z}$, then

$$Mv = \begin{bmatrix} 0 \\ \beta - b \\ 1 \end{bmatrix}.$$

Use

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

to permute the entries. Then

$$HMv = \begin{bmatrix} \beta - b \\ 1 \\ 0 \end{bmatrix}.$$

If $\beta \notin \mathbb{Z}$, then

$$\frac{1}{\beta - b} HMv = \begin{bmatrix} 1 \\ 1/(\beta - b) \\ 0 \end{bmatrix}.$$

If $\beta \in \mathbb{Z}$, apply H once more to obtain

$$H^2 Mv = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Repeating this process yields

$$v_{n+1} = (1, \alpha_{n+1}, \beta_{n+1})^T = \begin{cases} \left(1, \dfrac{\beta_n - b_n}{\alpha_n - a_n}, \dfrac{1}{\alpha_n - a_n}\right)^T, & \alpha_n \notin \mathbb{Z}, \\ \left(1, \dfrac{1}{\beta_n - b_n}, 0\right)^T, & \alpha_n \in \mathbb{Z},\ \beta_n \notin \mathbb{Z}, \\ (1, 0, 0)^T, & \alpha_n, \beta_n \in \mathbb{Z}, \end{cases}$$

where

$$a_n = \lfloor \alpha_n \rfloor, \qquad b_n = \lfloor \beta_n \rfloor, \qquad v_0 = (1, \alpha_0, \beta_0)^T = (1, \alpha, \beta)^T.$$

We also define

$$w_n = (1, \lfloor \alpha_n \rfloor, \lfloor \beta_n \rfloor)^T = (1, a_n, b_n)^T.$$

The vectors w_n always have integer entries. Since continued fractions use integer entries, this is the sequence we follow for higher-dimensional analogs. For a deeper analysis of products of these companion matrices and of this vector form of the Jacobi–Perron algorithm, see [6].

If the process reaches $(1, 0, 0)^T$, it remains there. We therefore terminate the sequence at that point; otherwise, the process continues indefinitely.

Example 2. Let $\alpha_0 = \sqrt[3]{2}$ and $\beta_0 = \sqrt{3}$. Then the first terms are

$$v_n: \quad \begin{bmatrix} 1 \\ \sqrt[3]{2} \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} 1 \\ (\sqrt{3} - 1)/(\sqrt[3]{2} - 1) \\ 1/(\sqrt[3]{2} - 1) \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ (4 - 3\sqrt[3]{2})/(\sqrt{3} - 2\sqrt[3]{2} + 1) \\ (\sqrt[3]{2} - 1)/(\sqrt{3} - 2\sqrt[3]{2} + 1) \end{bmatrix}, \ldots,$$

and

$$w_n: \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 26 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, \ldots.$$

The algorithm does not terminate because this choice of α_0 and β_0 gives $\alpha_n, \beta_n \notin \mathbb{Z}$ for every n. It expresses α_0 and β_0 as the ternary continued fractions

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1 + \cfrac{1}{26 + \cdots}}{1 + \cfrac{5}{26 + \cdots}}}$$

and

$$\sqrt{3} = 1 + \cfrac{2 + \cfrac{1}{1 + \cfrac{5}{26 + \cdots}}}{3 + \cfrac{1 + \cfrac{1}{26 + \cdots}}{1 + \cfrac{5}{26 + \cdots}}}.$$

Information about ternary continued fractions arising from the Jacobi–Perron algorithm can be found in [19,21].

Periodicity in a multidimensional continued fraction occurs when a block of w_i vectors repeats. The construction generalizes to $k \times k$ matrices by expanding M so that M^{-1} remains a companion matrix with nonnegative integer entries. The algorithm then has the form

$$v_{n+1} = (1, \alpha_{1,n+1}, \alpha_{2,n+1}, \ldots, \alpha_{k-1,n+1})^T,$$

where

$$\alpha_{j,n+1} = \frac{\alpha_{j,n} - a_{j,n}}{\alpha_{p,n} - a_{p,n}},$$

and p is the smallest index for which $\alpha_{p,n} \notin \mathbb{Z}$. If no such p exists, then

$$v_{n+1} = (1, 0, 0, \ldots, 0)^T$$

and the algorithm stops. As in the 2×2 case, products of the companion matrices record the multidimensional continued fraction. These products will be used to connect multidimensional continued fractions and Riordan arrays.

4 Connections Between Continued Fractions and Riordan Arrays

We now examine connections between Riordan arrays and matrix representations of ordinary continued fractions before turning to multidimensional continued fractions. We begin with recurrences obtained from eigenvalues and eigenvectors in the 2×2 case.

Suppose $[k_0; k_1 : k_2 : \cdots]$ is represented by the companion matrices, and let $[1, a]^T$ be an eigenvector of a product of these matrices. For one matrix,

$$\begin{bmatrix} 0 & 1 \\ 1 & k_0 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} a \\ 1 + k_0 a \end{bmatrix} = a \begin{bmatrix} 1 \\ k_0 + 1/a \end{bmatrix} = \lambda_0 \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

Thus, $\lambda_0 = a$ and

$$a = k_0 + \frac{1}{a}, \qquad a^2 - k_0 a - 1 = 0, \qquad a = \lambda_0 = \frac{k_0 \pm \sqrt{k_0^2 + 4}}{2}.$$

For two matrices,

$$\begin{bmatrix} 0 & 1 \\ 1 & k_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & k_1 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = (1 + k_1 a) \begin{bmatrix} 1 \\ k_0 + \dfrac{a}{1 + k_1 a} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

Hence,

$$\lambda_1 = 1 + k_1 a, \qquad a = k_0 + \frac{a}{1 + k_1 a} = k_0 + \frac{\lambda_0}{\lambda_1}.$$

Continuing, Proposition 2 gives

$$\begin{aligned} \left(\prod_{j=0}^{n} \begin{bmatrix} 0 & 1 \\ 1 & k_j \end{bmatrix} \right) \begin{bmatrix} 1 \\ a \end{bmatrix} &= \begin{bmatrix} Q_{n-1}(k_0, \ldots, k_{n-1}) + a Q_n(k_0, \ldots, k_n) \\ P_{n-1}(k_0, \ldots, k_{n-1}) + a P_n(k_0, \ldots, k_n) \end{bmatrix} \\ &= \big(Q_{n-1}(k_0, \ldots, k_{n-1}) + a Q_n(k_0, \ldots, k_n) \big) \begin{bmatrix} 1 \\ k_0 + \dfrac{Q_{n-2}(k_1, \ldots, k_{n-1}) + a Q_{n-1}(k_1, \ldots, k_n)}{Q_{n-1}(k_0, \ldots, k_{n-1}) + a Q_n(k_0, \ldots, k_n)} \end{bmatrix}. \end{aligned}$$

Although the convergents describe the entries, eigenvalues, and eigenvectors at each step, the notation is cumbersome. We therefore analyze period 1, so $k_j = k \in \mathbb{Z}_{>0}$ for all $j \geq 0$.

Let

$$A_k = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}, \qquad A_k^{n+1} \begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \end{bmatrix} = \lambda_n \begin{bmatrix} 1 \\ s_n \end{bmatrix}.$$

Set $r_1^{(-1)} = 1$, $r_2^{(-1)} = a$, $\lambda_{-1} = 1$, and $\lambda_0 = a$. Matrix multiplication gives

$$\lambda_{n+1} = r_2^{(n)} = r_1^{(n+1)},$$

$$a = s_{n+1} = k + \frac{\lambda_n}{\lambda_{n+1}} = k + \frac{r_1^{(n)}}{r_2^{(n)}} = k + \frac{r_1^{(n)}}{r_1^{(n+1)}},$$

and

$$r_1^{(n+1)} = r_2^{(n)}, \qquad r_2^{(n+1)} = kr_2^{(n)} + r_1^{(n)}.$$

To retain the stated initial values, define the shifted generating functions

$$R_1(t) = \sum_{n\geq 0} r_1^{(n-1)} t^n, \qquad R_2(t) = \sum_{n\geq 0} r_2^{(n-1)} t^n.$$

Then

$$\begin{aligned} R_1(t) &= 1 + \sum_{n\geq 1} r_1^{(n-1)} t^n = 1 + t \sum_{n\geq 0} r_1^{(n)} t^n \\ &= 1 + t \sum_{n\geq 0} r_2^{(n-1)} t^n = 1 + tR_2(t), \end{aligned}$$

and

$$\begin{aligned} R_2(t) &= a + \sum_{n\geq 1} r_2^{(n-1)} t^n = a + t \sum_{n\geq 0} r_2^{(n)} t^n \\ &= a + kt \sum_{n\geq 0} r_2^{(n-1)} t^n + t \sum_{n\geq 0} r_1^{(n-1)} t^n \\ &= a + ktR_2(t) + tR_1(t). \end{aligned}$$

Substitution gives

$$R_2(t) = \frac{t + a}{1 - kt - t^2}, \tag{2}$$

and consequently

$$R_1(t) = \frac{(a - k)t + 1}{1 - kt - t^2}. \tag{3}$$

If a and k are positive integers, these generating functions produce versions of the k-Fibonacci numbers. The Fibonacci recurrence is

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \qquad F_0 = 0, \quad F_1 = 1.$$

The k-Fibonacci sequence is defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad (n \geq 2), \qquad F_{k,0} = 0, \quad F_{k,1} = 1.$$

Properties of these numbers are given in [7]. We use the generating functions

$$\sum_{n=0}^{\infty} F_{k,n} t^n = \frac{t}{1 - kt - t^2} \tag{4}$$

and

$$\sum_{n=0}^{\infty} F_{k,n+1} t^n = \frac{1}{1 - kt - t^2}. \tag{5}$$

Theorem 2. *The coefficient sequences of $R_1(t)$ and $R_2(t)$ satisfy the k-Fibonacci recurrence. If $F_{k,n}^{(1)}$ and $F_{k,n}^{(2)}$ denote these coefficient sequences, respectively, then their initial values are*

$$F_{k,0}^{(1)} = 1, \qquad F_{k,1}^{(1)} = a, \qquad \text{and} \qquad F_{k,0}^{(2)} = a, \qquad F_{k,1}^{(2)} = 1 + ka,$$

where $k > 0$.

Proof. Equation (3) can be written as

$$R_1(t) = (a-k)\frac{t}{1-kt-t^2} + \frac{1}{1-kt-t^2} = (a-k)M(t) + N(t),$$

where $M(t)$ and $N(t)$ are the generating functions in equations (4) and (5). Since both coefficient sequences satisfy the same recurrence, every linear combination does also. The first two coefficients of $R_1(t)$ are

$$(a-k)(0) + 1 = 1, \qquad (a-k)(1) + k = a.$$

Similarly,

$$R_2(t) = \frac{t}{1-kt-t^2} + a\frac{1}{1-kt-t^2} = M(t) + aN(t),$$

whose first two coefficients are

$$0 + a = a, \qquad 1 + ak.$$

$\square$

Thus, the eigenvalue sequence follows a k-Fibonacci pattern, and the value s_n in the eigenvector is a ratio of k-Fibonacci numbers. This also connects k-Fibonacci numbers with purely periodic continued fractions of period 1.

We now connect $R_1(t)$ and $R_2(t)$ to Riordan arrays.

Theorem 3. *The generating functions $R_1(t)$ and $R_2(t)$ are the row-sum generating functions of*

$$\mathcal{R}\big((a-k)t+1, kt+t^2\big) \quad \textit{and} \quad \mathcal{R}\big(t+a, kt+t^2\big),$$

respectively, provided $a, k \neq 0$.

Proof. We have

$$R_1(t) = \frac{(a-k)t+1}{1-(kt+t^2)}, \qquad R_2(t) = \frac{t+a}{1-(kt+t^2)}.$$

Equation (1) gives the result. The condition $a \neq 0$ ensures that the second array satisfies $d(0) \neq 0$, and $k \neq 0$ ensures $h'(0) \neq 0$ for both arrays. $\square$

Theorem 3 does not describe all Riordan arrays having row sums represented by $R_1(t)$ and $R_2(t)$.

Theorem 4. *The generating functions $R_1(t)$ and $R_2(t)$ are also the row-sum generating functions of*

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{(a-1)t+(1+k-a)t^2}{1+(a-k-1)t-(a-k)t^2}\right)$$

and

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{(a-1)+(k+1-a)t}{a-(a-1)t-t^2}\right),$$

respectively, provided $a \neq 1$ in the first case, and $a = 1$ and $k \neq 0$ in the second case.

Proof. Using $d(t) = 1/(1-t)$, we rewrite

$$\begin{aligned} R_1(t) &= \frac{(a-k)t+1}{1-kt-t^2} \\ &= \frac{1}{1-t}\cdot\frac{1+(a-k-1)t-(a-k)t^2}{1-kt-t^2} \\ &= \frac{1}{1-t}\cdot\frac{1}{1-\dfrac{(a-1)t+(1+k-a)t^2}{1+(a-k-1)t-(a-k)t^2}}. \end{aligned}$$

Thus, $R_1(t)$ is the row-sum generating function of the first array. Its second component has $h(0) = 0$, and

$$h'(0) = a-1,$$

so $a \neq 1$ is necessary.

Similarly,

$$R_2(t) = \frac{1}{1-t}\cdot\frac{1}{1-\dfrac{(a-1)+(k+1-a)t}{a-(a-1)t-t^2}}.$$

For the second component to satisfy $h(0) = 0$, we must have $a = 1$. Under this restriction,

$$h(t) = \frac{kt}{1-t^2}, \qquad h'(0) = k,$$

so $k \neq 0$. $\square$

Theorems 3 and 4 show that $R_1(t)$ and $R_2(t)$ are row-sum generating functions for different Riordan arrays. These are only two possible constructions, so each generating function corresponds to a family rather than a unique Riordan array.

Example 3. Let $a = 2$ and $k = 1$. Theorems 3 and 4 give

$$\mathcal{R}(t+1, t+t^2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t^2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Both have row-sum sequence

$$1, 2, 3, 5, 8, \ldots,$$

a shifted Fibonacci sequence.

Example 4. Let $a = 1$ and $k = 2$. Then

$$\mathcal{R}(t+1, 2t+t^2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 4 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 8 & 8 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 20 & 16 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{2t}{1-t^2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 4 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 4 & 8 & 0 & 0 & \cdots \\ 1 & 4 & 12 & 8 & 16 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Their common row-sum sequence is

$$1, 3, 7, 17, 41, \ldots.$$

5 Connections Between Multidimensional Continued Fractions and Riordan Arrays

We now examine recurrences arising from the 3×3 matrix representation of multidimensional continued fractions. Suppose $\{k_n\}$ and $\{m_n\}$ are the sequences represented in the companion matrices, and let

$$\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$

be a generic eigenvector. For one matrix,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & k_0 \\ 0 & 1 & m_0 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 1+k_0b \\ a+m_0b \end{bmatrix} = b \begin{bmatrix} 1 \\ k_0+1/b \\ m_0+a/b \end{bmatrix} = \lambda_0 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}.$$

Thus, $\lambda_0 = b$,

$$a = k_0 + \frac{1}{\lambda_0}, \qquad b = m_0 + \frac{a}{b},$$

and therefore

$$\lambda_0^2 = m_0\lambda_0 + a, \qquad \lambda_0^2 - m_0\lambda_0 - a = 0.$$

Consequently,

$$b = \lambda_0 = \frac{m_0 \pm \sqrt{m_0^2+4a}}{2}.$$

For two companion matrices,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & k_0 \\ 0 & 1 & m_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & k_1 \\ 0 & 1 & m_1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$
$$= (a+bm_1) \begin{bmatrix} 1 \\ k_0 + \dfrac{b}{a+bm_1} \\ m_0 + \dfrac{1+bk_1}{a+bm_1} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}.$$

Hence,

$$\lambda_1 = a + bm_1, \qquad a = k_0 + \frac{b}{a+bm_1} = k_0 + \frac{\lambda_0}{\lambda_1},$$

and

$$b = m_0 + \frac{1+bk_1}{a+bm_1} = m_0 + \frac{1+bk_1}{\lambda_1}.$$

Multiplying three companion matrices gives

$$\left(\prod_{i=0}^{2} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & k_i \\ 0 & 1 & m_i \end{bmatrix} \right) \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$
$$= \big(1 + am_1 + b(k_2 + m_2m_1)\big) \begin{bmatrix} 1 \\ k_0 + \dfrac{a+bm_2}{1+am_1+b(k_2+m_2m_1)} \\ m_0 + \dfrac{ak_1 + b(1+m_2k_1)}{1+am_1+b(k_2+m_2m_1)} \end{bmatrix}.$$

Thus, the eigenvalue of the three-matrix product is

$$\lambda_2 = 1 + am_1 + b(k_2 + m_2m_1).$$

For $n+1$ companion matrices, write

$$\left(\prod_{i=0}^{n} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & k_i \\ 0 & 1 & m_i \end{bmatrix} \right) \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \\ r_3^{(n)} \end{bmatrix} = \lambda_n \begin{bmatrix} 1 \\ s_1^{(n)} \\ s_2^{(n)} \end{bmatrix}.$$

As in the 2×2 case, the general recurrences are unwieldy. We therefore assume period 1: $k_i = k \in \mathbb{Z}$ and $m_i = m \in \mathbb{Z}$ for all $i \geq 0$. Set

$$r_1^{(-1)} = 1, \qquad r_2^{(-1)} = a, \qquad r_3^{(-1)} = b, \qquad \lambda_{-1} = 1, \qquad \lambda_0 = b.$$

Matrix multiplication gives

$$\lambda_{n+1} = r_3^{(n)} = r_1^{(n+1)},$$

$$a = s_1^{(n+1)} = k + \frac{\lambda_n}{\lambda_{n+1}} = k + \frac{r_1^{(n)}}{r_3^{(n)}} = k + \frac{r_1^{(n)}}{r_1^{(n+1)}},$$

$$b = s_2^{(n+1)} = m + \frac{r_2^{(n)}}{r_3^{(n)}} = m + \frac{r_2^{(n)}}{r_1^{(n+1)}},$$

and

$$r_1^{(n+1)} = r_3^{(n)}, \qquad r_2^{(n+1)} = r_1^{(n)} + kr_3^{(n)}, \qquad r_3^{(n+1)} = r_2^{(n)} + mr_3^{(n)}.$$

Define the shifted generating functions

$$R_i(t) = \sum_{n\geq 0} r_i^{(n-1)} t^n, \qquad i = 1, 2, 3.$$

Then

$$R_1(t) = 1 + tR_3(t),$$

$$R_2(t) = a + tR_1(t) + ktR_3(t),$$

and

$$R_3(t) = b + tR_2(t) + mtR_3(t).$$

Solving this system gives

$$R_1(t) = \frac{(a-k)t^2 + (b-m)t + 1}{1 - mt - kt^2 - t^3},$$

$$R_2(t) = \frac{(b-m)t^2 + (1 + kb - ma)t + a}{1 - mt - kt^2 - t^3},$$

and

$$R_3(t) = \frac{t^2 + at + b}{1 - mt - kt^2 - t^3}.$$

These generating functions correspond to the recurrence

$$A_{k,m,n+1} = mA_{k,m,n} + kA_{k,m,n-1} + A_{k,m,n-2}, \qquad n \geq 2, \tag{6}$$

with $k, m \in \mathbb{Z}$ and initial values $A_{k,m,0}, A_{k,m,1}, A_{k,m,2} \in \mathbb{Z}$. This recurrence combines features of k-Fibonacci and generalized Pell recurrences. The generalized Pell numbers in [8] satisfy

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)},$$

with the stated initial conditions. More generally, one might consider

$$G_{n+1} = k_1 G_n + k_2 G_{n-1} + \cdots + k_j G_{n-j+1} + G_{n-j}. \tag{7}$$

For the moment, we restrict attention to equation (6).

Proposition 3. *Let $k, m \in \mathbb{Z}$, and let $A_{k,m,n}$ satisfy equation (6). Then*

$$\sum_{n\geq 0} A_{k,m,n} t^n = \frac{A_{k,m,0} + (A_{k,m,1} - mA_{k,m,0})t + (A_{k,m,2} - mA_{k,m,1} - kA_{k,m,0})t^2}{1 - mt - kt^2 - t^3}. \tag{8}$$

The proposition follows from the standard generating-function method described in Chapter 3 of [1].

Theorem 5. *The generating functions $R_1(t)$, $R_2(t)$, and $R_3(t)$ satisfy equation (6) with the following initial values:*

	$A_{k,m,0}$	$A_{k,m,1}$	$A_{k,m,2}$
$R_1(t)$	1	b	$a + mb$
$R_2(t)$	a	$1 + kb$	$b + ka + kmb$
$R_3(t)$	b	$a + mb$	$1 + m(a + mb) + kb.$

Proof. Substituting the displayed initial values into equation (8) gives the numerators of $R_1(t)$, $R_2(t)$, and $R_3(t)$, respectively. □

Example 5. When $k = a = 1$ and $m = b = 2$, the generating functions $R_1(t)$ and $R_3(t)$ produce shifted versions of the 3-Pell sequence.

We can now connect these generating functions to row sums of Riordan arrays.

Theorem 6. *The generating functions $R_1(t)$, $R_2(t)$, and $R_3(t)$ are the row-sum generating functions of*

$$\mathcal{R}\big((a-k)t^2+(b-m)t+1,\ mt+kt^2+t^3\big),$$

$$\mathcal{R}\big((b-m)t^2+(1+kb-ma)t+a,\ mt+kt^2+t^3\big),$$

and

$$\mathcal{R}\big(t^2+at+b,\ mt+kt^2+t^3\big),$$

respectively, provided $a, b, m \neq 0$.

Proof. Equation (1), together with the same rewriting used in the proof of Theorem 3, gives the result. The restrictions $a \neq 0$ and $b \neq 0$ ensure $d(0) \neq 0$ for the second and third arrays, and $m \neq 0$ ensures $h'(0) \neq 0$ for all three. □

Theorem 7. *The generating functions $R_1(t)$, $R_2(t)$, and $R_3(t)$ are also the row-sum generating functions of*

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{(b-1)t+(a-b+m)t^2+(k-a+1)t^3}{1+(b-m-1)t+(a-b-k+m)t^2-(a-k)t^3}\right),$$

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{bkt+(k-bk+b-1)t^2+(1-b+m)t^3}{1+(bk-m)t+(b-1-bk)t^2+(m-b)t^3}\right),$$

and

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{(m+a-1)t+(k+1-a)t^2}{1+(a-1)t+(1-a)t^2-t^3}\right),$$

respectively, provided

$$b \neq 1; \qquad a = 1 \text{ and } b, k \neq 0; \qquad \text{and} \qquad b = 1 \text{ and } a+m \neq 1,$$

respectively.

Proof. As in Theorem 4, use algebraic manipulation to place each generating function in the form of equation (1) with $d(t) = 1/(1-t)$.

For $R_1(t)$,

$$\begin{aligned} R_1(t) &= \frac{(a-k)t^2+(b-m)t+1}{1-mt-kt^2-t^3} \\ &= \frac{1}{1-t} \cdot \frac{1+(b-m-1)t+(a-b-k+m)t^2-(a-k)t^3}{1-mt-kt^2-t^3} \\ &= \frac{1}{1-t} \cdot \frac{1}{1-\dfrac{(b-1)t+(a-b+m)t^2+(k-a+1)t^3}{1+(b-m-1)t+(a-b-k+m)t^2-(a-k)t^3}}. \end{aligned}$$

The second component has $h(0) = 0$, and $h'(0) = b-1$, so $b \neq 1$.

For $R_2(t)$, before imposing restrictions, write

$$R_2(t) = \frac{1}{1-t} \cdot \frac{1}{1-H(t)},$$

where $H(t) = y(t)/z(t)$ with

$$\begin{aligned} y(t) = (a-1) &+ (m+1-a+bk-am)t \\ &+ (k+am-bk+b-m-1)t^2 + (1-b+m)t^3, \end{aligned}$$

and

$$\begin{aligned} z(t) = a &+ (1-a+bk-am)t \\ &+ (b-m-1+am-bk)t^2 + (m-b)t^3. \end{aligned}$$

The condition $H(0) = 0$ forces $a = 1$. Under that restriction,

$$H(t) = \frac{bkt+(k-bk+b-1)t^2+(1-b+m)t^3}{1+(bk-m)t+(b-1-bk)t^2+(m-b)t^3},$$

and $H'(0) = bk$, so $b, k \neq 0$.

Finally,

$$R_3(t) = \frac{1}{1-t} \cdot \frac{1}{1 - \dfrac{(b-1) + (m+a-b)t + (k+1-a)t^2}{b + (a-b)t + (1-a)t^2 - t^3}}.$$

The condition $h(0) = 0$ forces $b = 1$. The second component then becomes

$$\frac{(m+a-1)t + (k+1-a)t^2}{1 + (a-1)t + (1-a)t^2 - t^3},$$

whose derivative at zero is $a + m - 1$. Thus, $a + m \neq 1$. $\square$

Example 6. Consider period 1 with $a = b = 1$ and $k = m = 2$. Theorem 6 gives the arrays

$$\mathcal{R}(-t^2 - t + 1,\ 2t + 2t^2 + t^3),$$

$$\mathcal{R}(-t^2 + t + 1,\ 2t + 2t^2 + t^3),$$

and

$$\mathcal{R}(t^2 + t + 1,\ 2t + 2t^2 + t^3).$$

Theorem 7 gives, respectively,

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{2t^2 + 2t^3}{1 - 2t + t^3}\right),$$

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{2t + 2t^3}{1 - 2t^2 + t^3}\right),$$

and

$$\mathcal{R}\left(\frac{1}{1-t}, \frac{2t + 2t^2}{1 - t^3}\right).$$

Thus,

$$\mathcal{R}(-t^2 - t + 1,\ 2t + 2t^2 + t^3) \quad \text{and} \quad \mathcal{R}\left(\frac{1}{1-t}, \frac{2t^2 + 2t^3}{1 - 2t + t^3}\right)$$

have the common row-sum sequence

$$1, 1, 3, 9, 25, 71, 201, \ldots,$$

a shifted version of OEIS sequence A233828 [14]. Also,

$$\mathcal{R}(t^2 + t + 1,\ 2t + 2t^2 + t^3) \quad \text{and} \quad \mathcal{R}\left(\frac{1}{1-t}, \frac{2t + 2t^2}{1 - t^3}\right)$$

have the same row sums. Finally,

$$\mathcal{R}(-t^2 + t + 1,\ 2t + 2t^2 + t^3) \quad \text{and} \quad \mathcal{R}\left(\frac{1}{1-t}, \frac{2t + 2t^3}{1 - 2t^2 + t^3}\right)$$

have row-sum sequence

$$1, 3, 7, 21, 59, 167, 473, \ldots.$$

The same method extends to higher dimensions. In the $k \times k$ case, consider

$$\left(\prod_{i=0}^{n} \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & m_{1,i} \\ 0 & 1 & 0 & \cdots & m_{2,i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & m_{k-1,i} \end{bmatrix}\right) \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \\ \vdots \\ r_k^{(n)} \end{bmatrix} = \lambda_n \begin{bmatrix} 1 \\ s_1^{(n)} \\ \vdots \\ s_{k-1}^{(n)} \end{bmatrix}.$$

Assume period 1, so $m_{j,i} = m_j$ for $1 \leq j \leq k-1$ and $i \geq 0$. With initial values

$$\lambda_{-1} = 1, \qquad \lambda_0 = a_{k-1}, \qquad r_1^{(-1)} = 1, \qquad r_2^{(-1)} = a_1, \quad \ldots, \quad r_k^{(-1)} = a_{k-1},$$

we obtain

$$\lambda_{n+1} = r_k^{(n)} = r_1^{(n+1)},$$

$$a_j = s_j^{(n+1)} = m_j + \frac{r_{j+1}^{(n)}}{r_k^{(n)}} = m_j + \frac{r_{j+1}^{(n)}}{r_1^{(n+1)}}, \qquad 1 \leq j \leq k-1,$$

and

$$r_1^{(n+1)} = r_k^{(n)},$$
$$r_{j+1}^{(n+1)} = r_j^{(n)} + m_j r_k^{(n)}, \qquad 1 \le j \le k-1.$$

We conjecture that solving these recurrences yields

$$R_1(t) = \frac{(a_1 - m_1)t^{k-1} + (a_2 - m_2)t^{k-2} + \cdots + (a_{k-1} - m_{k-1})t + 1}{1 - m_{k-1}t - m_{k-2}t^2 - \cdots - m_1 t^{k-1} - t^k}$$

and

$$R_k(t) = \frac{t^{k-1} + a_1 t^{k-2} + \cdots + a_{k-2}t + a_{k-1}}{1 - m_{k-1}t - m_{k-2}t^2 - \cdots - m_1 t^{k-1} - t^k}.$$

We further conjecture that these recurrences produce generating functions for the generalized recurrence in equation (7). Although k Riordan arrays are associated with each $k \times k$ system, the arrays associated with $R_1(t)$ and $R_k(t)$ dominate in the sense that their row sums are shifted versions of the same sequence. It would be interesting to determine whether these two generating functions suffice to classify certain periodic multidimensional continued fractions.

6 Future Directions

We have exhibited a new connection between multidimensional continued fractions and families of Riordan arrays. These connections rely on purely periodic multidimensional continued fractions of period 1. One direction for future work is to study non-purely-periodic cases. Such cases alter the initial conditions and parts of the calculation, and consequently change the generating functions that connect the multidimensional continued fractions to families of Riordan arrays.

A second direction is to consider periods greater than 1. Even period 2 introduces an additional layer of complexity because odd- and even-indexed terms must be treated separately. Can these results be generalized to arbitrary periods? The connections between multidimensional continued fractions and Riordan arrays may provide a way to identify additional n-tuples that produce periodic multidimensional continued fractions. The associated families of Riordan arrays also deserve further study. What can they reveal about Riordan arrays that have the same row sums, or about periodic multidimensional continued fractions?

Acknowledgements: Since this work was developed as part of the author's doctoral dissertation, the author thanks her advisor, Matt Hudelson, for his help and guidance.

References

[1] M. Aigner, *A Course in Enumeration*, Springer, 2007.

[2] P. Arnoux, V. Berthé, and S. Ito, *Discrete planes, $\mathbb{Z}^2$-actions, Jacobi–Perron algorithm and substitutions*, Ann. Inst. Fourier (Grenoble) 52 (2002), 305–349.

[3] P. Barry, *Continued fractions and transformations of integer sequences*, J. Integer Seq. 12 (2009), Article 09.7.6.

[4] P. Barry, *Riordan pseudo-involutions, continued fractions and Somos-4 sequences*, arXiv:1807.05794, 2018.

[5] P. Barry, *Riordan arrays and Jacobi and Thron continued fractions*, arXiv:2109.07333, 2021.

[6] L. Bernstein, *The Jacobi–Perron Algorithm: Its Theory and Application*, Lecture Notes in Mathematics, Vol. 207, Springer-Verlag, 1971.

[7] C. Bolat and H. Köse, *On the properties of k-Fibonacci numbers*, Int. J. Contemp. Math. Sci. 5 (2010), 1097–1105.

[8] J. J. Bravo, J. L. Herrera, and J. L. Ramírez, *Combinatorial interpretation of generalized Pell numbers*, J. Integer Seq. 23 (2020), Article 20.2.1.

[9] K. Dasaratha, L. Flapan, T. Garrity, C. Lee, C. Mihaila, N. Neumann-Chun, S. Peluse, and M. Stoffregen, *Cubic irrationals and periodicity via a family of multidimensional continued-fraction algorithms*, Monatsh. Math. 174 (2014), 549–566.

[10] D. Dumont, *Further triangles of Seidel–Arnold type and continued fractions related to Euler and Springer numbers*, Adv. in Appl. Math. 16 (1995), 275–296.

[11] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. 32 (1980), 125–161.

[12] A. Hennessy, *A Study of Riordan Arrays with Applications to Continued Fractions, Orthogonal Polynomials and Lattice Paths*, Ph.D. thesis, Waterford Institute of Technology, 2011.

[13] M. Hudelson, Private communication.

[14] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org/`.

[15] M. E. H. Ismail and D. R. Masson, *Generalized orthogonality and continued fractions*, J. Approx. Theory 83 (1995), 1–40.

[16] S. Ito and M. Ohtsuki, *Parallelogram tilings and Jacobi–Perron algorithm*, Tokyo J. Math. 17 (1994), 33–58.

[17] O. Karpenkov, *Geometry of Continued Fractions*, Algorithms and Computation in Mathematics, Vol. 26, Springer, 2013.

[18] O. Karpenkov, *On Hermite's problem, Jacobi–Perron type algorithms, and Dirichlet groups*, arXiv:2101.12707, 2021.

[19] J. T. Kent, *Ternary continued fractions and the evenly tempered musical scale*, CWI Newsletter 13 (1986), 21–33.

[20] A. Luzón, D. Merlini, M. A. Morón, and R. Sprugnoli, *Identities induced by Riordan arrays*, Linear Algebra Appl. 436 (2012), 631–647.

[21] N. Murru, *Linear recurrence sequences and periodicity of multidimensional continued fractions*, Ramanujan J. 44 (2017), 115–124.

[22] R. Perrier, *Using Higher Dimensional Extensions of TRIP Maps for Analyzing Periodicity of Multidimensional Continued Fractions*, Ph.D. thesis, Washington State University, 2023.

[23] F. Schweiger, *The Metrical Theory of Jacobi–Perron Algorithm*, Lecture Notes in Mathematics, Vol. 334, Springer-Verlag, 1973.

[24] F. Schweiger, *Multidimensional Continued Fractions*, Oxford University Press, 2000.

[25] L. Shapiro, R. Sprugnoli, P. Barry, G.-S. Cheon, T.-X. He, D. Merlini, and W. Wang, *The Riordan Group and Applications*, Springer, 2022.

[26] L. W. Shapiro, S. Getu, W.-J. Woan, and L. C. Woodson, *The Riordan group*, Discrete Appl. Math. 34 (1991), 229–239.

[27] R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. 132 (1994), 267–290.

[28] H. S. Wilf, *Generatingfunctionology*, Academic Press, 1990.

[29] P. Wynn, *On some recent developments in the theory and application of continued fractions*, J. Soc. Indust. Appl. Math. Ser. B Numer. Anal. 1 (1964), 177–197.