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# Periodic Column Partial Sums in the Riordan Array of a Polynomial

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Abstract: When $p(t)$ is a polynomial of degree $d$, the $k$-th column of the Riordan array

$$\left(\frac{1}{1-t^{d+1}}, tp(t)\right)$$

is an eventually periodic sequence whose repeating part begins at the $\big(1+(k-1)(d+1)\big)$-st term. The preperiodic terms add up to the $(k-1)(d+1)$-st partial sum of the corresponding formal power series, and thus the Riordan array of $p(t)$ generates a sequence of column partial sums. We classify the linear and quadratic polynomials, and present a particular family of polynomials of higher degree, for which these sequences of column partial sums are eventually periodic.

**Keywords**: Circulant matrices; Generating functions; Partial sums of a series; Riordan arrays
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# 1 Introduction

Let

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d \in \mathbb{C}[t]$$

be a polynomial of degree $d \geq 0$ with $a_0 \neq 0$. Taking coefficients of the formal power series

$$\frac{p(t)}{1-t^{d+1}} = \frac{a_0 + a_1 t + \cdots + a_d t^d}{1-t^{d+1}} \in \mathbb{C}[[t]],$$

we obtain an infinite periodic sequence with repeating block $\{a_0, a_1, \ldots, a_d\}$. It is easy to see that for every $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the coefficients of the formal power series

$$\frac{(tp(t))^k}{1-t^{d+1}} \tag{1.1}$$

also form an eventually periodic sequence [6, Theorem 1]. We can put these sequences together into a lower-triangular matrix whose $k$-th column, for $k \geq 0$, is the coefficient sequence of (1.1). This matrix is an example of a Riordan array, which is defined by a pair of formal power series $(f(t), g(t))$. In the example above,

$$f(t) = \frac{1}{1-t^{d+1}}, \qquad g(t) = tp(t).$$

In general, a proper Riordan array over a field $F$ is defined by a pair of formal power series $(f(t), g(t))$, where

$$f(t) = \sum_{i\geq 0} f_i t^i \in F[[t]], \qquad g(t) = \sum_{i\geq 0} g_i t^i \in F[[t]].$$

To guarantee nonzero entries along the main diagonal and the lower-triangular form of the array, the order of $f(t)$ must be 0, so $f_0 \neq 0$, and the order of $g(t)$ must be 1, so $g_0 = 0$ and $g_1 \neq 0$.

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We are interested in certain properties of the Riordan arrays

$$(f(t), g(t)) = \left(\frac{1}{1-t^{d+1}}, tp(t)\right), \tag{1.2}$$

where $p(t)$ is a polynomial of degree $d \geq 0$ with complex coefficients and $p(0) \neq 0$. We call such an array the *Riordan array of* $p(t)$ and abbreviate it by $\mathrm{RA}_p$. For the general theory of Riordan arrays and the Riordan group, we refer the reader to the original paper [8], where this terminology was introduced, the introductory text [1], and the comprehensive monograph [9]. The latter is an excellent resource on recent developments and the corresponding literature. For brief introductions to the topic, see the survey articles [3, 4].

More specifically, we continue the study of the Riordan arrays (1.2) begun in [6], focusing on polynomials $p(t)$ that generate circulant matrices of finite period, or finite order when the matrix is nonsingular. We follow standard terminology from discrete dynamical systems. Let $f : X \to X$ be a map describing the time evolution of points of a set $X$. If $x \in X$, consider the iterates

$$x, f(x), f(f(x)), \ldots,$$

and denote the $n$-th iterate of $f$ at $x$ by $f^n(x)$, with the convention that $f^0 = I$ is the identity map. The forward orbit of $x$ under the iterates of $f$ is

$$\mathcal{O}_f^+(x) := \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\} = \{f^n(x) : n \in \mathbb{N}_0\}.$$

A point $x \in X$ is called *preperiodic with period* $\mu \in \mathbb{N}$ if its orbit becomes periodic with period $\mu$ after finitely many steps; that is, if there exists $k \in \mathbb{N}_0$ such that

$$f^{\mu+j}(f^k(x)) = f^j(f^k(x)) \qquad \text{for all } j \in \mathbb{N}_0.$$

In this case, we also say that the orbit $\mathcal{O}_f^+(x)$ is eventually periodic. If $k = 0$, then $x$ is called periodic. Clearly, a point $x$ is preperiodic if and only if $\mathcal{O}_f^+(x)$ is finite.

For a polynomial

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d,$$

write its coefficients as the row vector

$$\nu = (a_d, a_{d-1}, \ldots, a_0) \in \mathbb{C}^{d+1},$$

and consider the shift operator $T : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}$ defined by

$$T(a_d, a_{d-1}, \ldots, a_1, a_0) = (a_0, a_d, a_{d-1}, \ldots, a_1).$$

Following [5], form the $(d+1) \times (d+1)$ matrix

$$V_{p(t)} = \mathrm{circ}\{\nu\},$$

called the circulant matrix associated with $p(t)$. Its rows are the iterates $T^i\nu$, $i \in \{0, \ldots, d\}$:

$$V_{p(t)} = \begin{pmatrix} a_d & a_{d-1} & \cdots & a_1 & a_0 \\ a_0 & a_d & \cdots & a_2 & a_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{d-2} & a_{d-3} & \cdots & a_d & a_{d-1} \\ a_{d-1} & a_{d-2} & \cdots & a_0 & a_d \end{pmatrix}.$$

If $d = 0$ and $p(t) = a_0$, then $V_{p(t)} = (a_0)$.

We consider the discrete dynamical system $f : X \to X$, where $X = \mathbb{C}^{d+1}$, the point is

$$x = (0, 0, \ldots, 0, 1)^T \in \mathbb{C}^{d+1},$$

and $f$ is the linear map represented by $V_{p(t)}$. Thus,

$$f^{\mu+j}(f^k(x)) = f^j(f^k(x)) \qquad \text{for some } \mu \geq 1,\ k \geq 0, \text{ and all } j \in \mathbb{N}_0. \tag{1.3}$$

To simplify terminology, we say that $V_{p(t)}$ is periodic with period $\mu$ if it satisfies (1.3), where $\mu$ is the smallest positive integer with this property.

For example, take $p(t) = \frac{1}{2} - \frac{1}{2}t$. Then

$$V_{p(t)} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad V_{p(t)}^2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad V_{p(t)}^3 = V_{p(t)}.$$

The orbit of $(0,1)^T$ is

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\},$$

and its eventual cycle has period 2.

According to [6, Theorem 2], each column of the Riordan array (1.2) is an eventually periodic sequence. With the top row indexed by $n = 0$ and the leftmost column indexed by 0, the periodic cycle of the $k$-th column, $k \geq 1$, begins at the $\big(1 + (k-1)(d+1)\big)$-st term and equals the $k$-th iterate of $V_{p(t)}$ at $(0, \ldots, 0, 1)^T$; see [6, Theorems 1 and 2]. For $p(t) = \frac{1}{2} - \frac{1}{2}t$, the array is

$$\left( \frac{1}{1-t^2}, \frac{t-t^2}{2} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{8} & \frac{1}{16} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{32} & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{7}{16} & -\frac{5}{32} & \frac{1}{64} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.4}$$

Polynomials that generate periodic $V_{p(t)}$ produce doubly periodic Riordan arrays: in addition to eventual periodicity in each column, there is horizontal repetition, as visible in (1.4). For some polynomials, the preperiodic terms in each column add to numbers that also vary periodically. For the example $p(t) = \frac{1}{2} - \frac{1}{2}t$, adding the first $2k-1$ terms, beginning with row 0, in the $k$-th column gives

$$\left\{ 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \ldots \right\}.$$

The $k$-th column of $\mathrm{RA}_p$ is the coefficient sequence of

$$\sum_{i=0}^{\infty} C_{i,k} t^i = \frac{(tp(t))^k}{1 - t^{d+1}}. \tag{1.5}$$

The sum of all entries in the preperiodic part of the $k$-th column, including the entry in row 0, is the $(k-1)(d+1)$-st partial sum

$$S_{(k-1)(d+1)} = \sum_{i=0}^{(k-1)(d+1)} C_{i,k}. \tag{1.6}$$

**Definition 1.** Given the Riordan array (1.2) of a degree-$d$ polynomial $p(t)$, the $(k-1)(d+1)$-st partial sum (1.6) of the formal power series (1.5) is denoted by $S_{[k]}$ and called the *partial sum of the k-th column*, or simply the *column partial sum*.

Here are two further examples from [6]. If

$$p(t) = \frac{-1 + 2t + 2t^2}{3},$$

then the sequence $\{S_{[k]}\}_{k \geq 1}$ has period 6; see [6, Proposition 5] or Proposition 18 below. Its repeating block is

$$\left\{ 0, -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}, 0 \right\}.$$

If

$$p(t) = \frac{2 - t + 2t^2}{3},$$

then the sequence of partial sums is not eventually periodic; its first terms are

$$\{S_{[k]}\}_{k \geq 1} = \left\{ 0, 0, \frac{1}{3}, 1, \frac{5}{3}, 2, 2, 2, \frac{7}{3}, 3, \ldots \right\}.$$

The main objective of this paper is to find polynomials $p(t) \in \mathbb{C}[t]$ for which the sequence

$$\{S_{[k]}\}_{k\geq 1} \tag{1.7}$$

is eventually periodic in the Riordan array (1.2). Linear polynomials with this property are classified in Theorem 11. A particular family of higher-degree polynomials is presented in Proposition 18. The following theorem classifies the quadratic case; its proof is divided among Propositions 13–15 in Section 3.

**Theorem 2.** *Let $p(t) = a + bt + ct^2 \in \mathbb{C}[t]$ with $ac \neq 0$. Let $\xi = e^{2\pi i/3}$ and let the eigenvalues of $V_{p(t)}$ be*

$$\lambda_1 = a + b + c, \qquad \lambda_2 = a\xi^2 + b\xi + c, \qquad \lambda_3 = a\xi + b\xi^2 + c.$$

*Then the following statements hold.*

- *If $\lambda_1 \neq 0$, then $V_{p(t)}$ has a finite period $\mu$ and the column partial sums are eventually periodic if and only if*
  $$p(t) = a(1 - 2t - 2t^2), \qquad (3a)^\mu = 1, \qquad 6 \mid \mu.$$
- *If $\lambda_1 = \lambda_2 = 0$, then $V_{p(t)}$ has a finite period $\mu$ and the column partial sums are eventually periodic, with*
  $$S_{[k]} = (3a\xi)^k \frac{\xi - 1}{9\xi}, \qquad k \geq 2,$$
  *if and only if*
  $$p(t) = a(1 - t)(1 - \xi t), \qquad (3a\xi)^\mu = 1.$$
- *If $\lambda_1 = \lambda_3 = 0$, then $V_{p(t)}$ has a finite period $\mu$ and the column partial sums are eventually periodic, with*
  $$S_{[k]} = (3a\xi^2)^k \frac{1 - \xi}{9}, \qquad k \geq 2,$$
  *if and only if*
  $$p(t) = a(1 - t)(\xi^2 - \xi t), \qquad (3a\xi^2)^\mu = 1.$$
- *If $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$, then $V_{p(t)}$ has a finite period $\mu$ and the column partial sums are eventually periodic if and only if*
  $$p(t) = a(1 - t)(1 + (r + 1)t),$$
  *where $a$ and $r$ satisfy*
  $$\begin{cases} \big((\xi - 1)a(r - \xi^2)\big)^\mu = 1, \\ \big((\xi - 1)a(1 - \xi^2 r)\big)^\mu = 1. \end{cases}$$
  *In this case, $r \in \mathbb{R}$ is a root of a polynomial of degree $\mu$ with rational coefficients.*

It is noteworthy that 1 is a root of every polynomial in the last three cases, where $\lambda_1 = 0$. In the first case, the roots of $p(t)$ are the irrational numbers $(-1 \pm \sqrt{3})/2$, independently of $a \neq 0$.

The paper is organized as follows. Section 2 contains technical lemmas, including a representation of the column partial sums as coefficients of formal power series. Section 3 contains the main results for linear and quadratic polynomials: Theorem 11 and Propositions 13–15. Section 4 proves eventual periodicity for the column partial sums of

$$p(t) = a\big((d - 1) - 2t - 2t^2 - \cdots - 2t^d\big),$$

when $d \geq 2$ and $a(d + 1)$ is a root of unity. Section 5 illustrates several eventually periodic column-partial-sum sequences by graphs in the complex plane.

Throughout the paper, if $p(t)$ has degree $d$, then

$$\xi = e^{2\pi i/(d+1)}$$

denotes a primitive $(d + 1)$-st root of unity, unless otherwise specified.

## 2 Riordan array of column partial sums and auxiliary lemmas

For a nonzero constant polynomial $p(t) = a$, the sequence of partial sums (1.7) consists entirely of zeros. This holds for every $a \in \mathbb{C} \setminus \{0\}$, since

$$\mathrm{RA}_a = \left(\frac{1}{1-t}, ta\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & a & 0 & 0 & 0 & \cdots \\ 1 & a & a^2 & 0 & 0 & \cdots \\ 1 & a & a^2 & a^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Henceforth, assume that $p(t)$ has degree $d \geq 1$, and denote the Riordan array (1.2) of $p(t)$ by $(f, g)$. If all partial sums in every column of $\mathrm{RA}_p$ are recorded, the resulting matrix is another Riordan array, namely

$$\left(\frac{1}{1-t}, t\right)\left(\frac{1}{1-t^{d+1}}, tp(t)\right) = \left(\frac{1}{(1-t)(1-t^{d+1})}, tp(t)\right). \tag{2.1}$$

Indeed, the $n$-th partial sum of the $m$-th column is obtained by multiplying that column by the infinite row vector with 1 in its first $n+1$ positions and 0 thereafter; compare [2, Proposition 3] and [9, Theorem 2.7].

Consequently, the sequence (1.7) consists of particular entries of the Riordan array (2.1). Since the $k$-th column partial sum contains the first $(k-1)(d+1)+1$ entries, we have

$$S_{[k]} = S_{(k-1)(d+1)} = \sum_{i=0}^{(k-1)(d+1)} C_{i,k}.$$

If the entries of (2.1) are denoted by $h_{n,k}$, then

$$\{S_{[k]}\}_{k\geq 1} = \{h_{0,1}, h_{d+1,2}, h_{2(d+1),3}, \ldots, h_{\ell(d+1),\ell+1}, \ldots\}.$$

Thus,

$$S_{[k]} = h_{(k-1)(d+1),k} = [t^{(k-1)(d+1)}]\frac{(tp(t))^k}{(1-t)(1-t^{d+1})}, \qquad k \geq 1. \tag{2.2}$$

In particular, $S_{[1]} = 0$ for every $p(t)$ with $p(0) \neq 0$.

It is well known that rational generating functions correspond to linear-recurrence sequences and conversely; see [7, Section 2.3]. We first record a characterization of eventually periodic sequences.

**Lemma 3.** *The generating function of a sequence $\{s_i\}_{i\geq 0}$ can be written as*

$$\sum_{i=0}^{\infty} s_i t^i = \frac{Q(t)}{1-t^n} \tag{2.3}$$

*for some polynomial $Q(t) \in \mathbb{C}[t]$ if and only if $\{s_i\}_{i\geq 0}$ is eventually periodic with a period dividing $n$.*

*Proof.* Suppose the periodicity starts at $s_k$ and the repeating block is

$$\{s_k, s_{k+1}, \ldots, s_{k+n-1}\}.$$

Set

$$q_0(t) = s_0 + s_1 t + \cdots + s_{k-1}t^{k-1}, \qquad q(t) = s_k + s_{k+1}t + \cdots + s_{k+n-1}t^{n-1}.$$

Then

$$\sum_{i=0}^{\infty} s_i t^i = q_0(t) + q(t)t^k + q(t)t^{k+n} + q(t)t^{k+2n} + \cdots = \frac{q_0(t)(1-t^n) + q(t)t^k}{1-t^n}.$$

Conversely, assume (2.3), and write

$$Q(t) = q_0 + q_1 t + \cdots + q_k t^k.$$

Then

$$\sum_{i=0}^{\infty} s_i t^i = \frac{Q(t)}{1-t^n} = \sum_{a=0}^{\infty}(q_0 + q_1 t + \cdots + q_k t^k)t^{na}. \tag{2.4}$$

If $k < n$, then the sequence is periodic with repeating block

$$\{q_0, q_1, \ldots, q_k, \underbrace{0, \ldots, 0}_{n-k-1}\}.$$

If $k \geq n$, write $k = pn + r$, where $0 \leq r < n$, and decompose

$$Q(t) = \sum_{i=0}^{p-1} Q_i(t)t^{in} + R(t)t^{pn},$$

where

$$Q_i(t) = q_{in} + q_{in+1}t + \cdots + q_{(i+1)n-1}t^{n-1}$$

and

$$R(t) = q_{pn} + q_{pn+1}t + \cdots + q_{pn+r}t^r.$$

Equation (2.4) becomes

$$\begin{aligned}\sum_{i=0}^{\infty} s_i t^i = Q_0(t) &+ \big(Q_0(t) + Q_1(t)\big)t^n + \cdots \\ &+ \big(Q_0(t) + \cdots + Q_{p-1}(t)\big)t^{(p-1)n} \\ &+ \big(Q_0(t) + \cdots + Q_{p-1}(t) + R(t)\big)\frac{t^{pn}}{1-t^n}.\end{aligned}$$

Thus the sequence is eventually periodic. If

$$\Pi(t) = Q_0(t) + \cdots + Q_{p-1}(t) + R(t) = p_0 + p_1 t + \cdots + p_{n-1}t^{n-1},$$

then the repeating block is the cyclic permutation

$$\{p_{r+1}, \ldots, p_{n-1}, p_0, \ldots, p_r\}.$$

□

The numerator and denominator in (2.3) need not be relatively prime. For example, if $\xi = e^{2\pi i/3}$, then the periodic sequence with repeating block $\{1, \xi, \xi^2\}$ has generating function

$$\frac{1 + \xi t + \xi^2 t^2}{1 - t^3} = \frac{\xi^2(1-t)(\xi - t)}{(1-t)(\xi - t)(\xi^2 - t)} = \frac{\xi^2}{\xi^2 - t} = \frac{1}{1 - \xi t}.$$

**Definition 4.** Let

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d \in \mathbb{C}[t]$$

have degree $d \geq 0$ and $a_0 \neq 0$, and let $V_{p(t)}$ be its associated circulant matrix. For each $k \in \mathbb{N}_0$, define

$$p_k(t) = \sum_{i=0}^{d} a_{k,i} t^i := (1, t, \ldots, t^d) V_{p(t)}^k \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{pmatrix}.$$

Until Section 4, we take $\xi = e^{2\pi i/3}$.

**Lemma 5.** *For every $k \in \mathbb{N}_0$,*

$$p_k(1) = \left(\sum_{i=0}^{d} a_i\right)^{k+1}.$$

*Proof.* The case $k = 0$ is immediate because $p_0(t) = p(t)$. Suppose the identity holds for $k$. Then

$$p_{k+1}(1) = (1, 1, \ldots, 1) V_{p(t)}^{k+1} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{pmatrix}$$

$$= \big((1,1,\dots,1)V_{p(t)}\big)\begin{pmatrix} a_{k,0}\\ a_{k,1}\\ \vdots\\ a_{k,d}\end{pmatrix}$$
$$= \left(\sum_{i=0}^{d} a_i\right)(a_{k,0}+a_{k,1}+\cdots+a_{k,d})$$
$$= \left(\sum_{i=0}^{d} a_i\right)\left(\sum_{i=0}^{d} a_i\right)^{k+1} = \left(\sum_{i=0}^{d} a_i\right)^{k+2}.$$

□

**Lemma 6.** *Suppose $z, w \in \mathbb{C}\setminus\{0\}$, $|w| = 1$, and*

$$|z-w| = |z-\xi w|.$$

*Then $z$ lies on the line through the origin that bisects the angle $2\pi/3$ between the vectors represented by $w$ and $\xi w$.*

*Proof.* Represent $z, w$, and $\xi w$ by points $A, B_1$, and $B_2$ in the complex plane, and denote the origin by $O$. Let

$$\alpha_i = \angle AOB_i, \qquad i \in \{1,2\}.$$

The hypotheses and the law of cosines imply $\cos\alpha_1 = \cos\alpha_2$. Choosing the angles in $[0,\pi]$ gives $\alpha_1 = \alpha_2$, so $OA$ lies on the required angle bisector. □

**Lemma 7.** *For every $k \geq 2$,*

$$[t^{2k-3}]\frac{(1-t^2)^{k-2}(1+t)^2}{1+t+t^2} = 2(-\sqrt{3})^{k-3}\sin\left(\frac{(k-4)\pi}{6}\right). \tag{2.5}$$

*Proof.* Using the binomial theorem and the power-series expansion of $(1+t)^2/(1+t+t^2)$, write

$$[t^{2k-3}]\frac{(1-t^2)^{k-2}(1+t)^2}{1+t+t^2} = [t^{2(k-1)}]\left((1-t^2)^{k-2}\frac{t(1+t)^2}{1+t+t^2}\right).$$

Since the binomial expansion contains only even powers, only the even-power terms in

$$\begin{aligned}\frac{t(1+t)^2}{1+t+t^2} &= t + \frac{t^2(1-t)}{1-t^3}\\ &= t + t^2(1-t)(1+t^3+t^6+\cdots)\\ &= t+t^2-t^3+t^5-t^6+t^8-t^9+t^{11}-t^{12}+\cdots\end{aligned}$$

contribute. Hence the left-hand side of (2.5) is

$$[t^{2(k-1)}]\sum_{i=0}^{k-2}(-1)^i\binom{k-2}{i}t^{2i}(t^2-t^6+t^8-t^{12}+t^{14}-t^{18}+\cdots). \tag{2.6}$$

Replacing $t^2$ by $x$ and using convolution, this becomes

$$\sum_{n=0}^{k-2}(-1)^n\binom{k-2}{n}[x^{k-1-n}](x-x^3+x^4-x^6+x^7-x^9+\cdots). \tag{2.7}$$

The coefficients of the series in (2.7) have repeating block $\{1,0,-1\}$. If $\eta = e^{2\pi i/6}$, then

$$\big\{\mathrm{Im}(\eta), \mathrm{Im}(\eta^3), \mathrm{Im}(\eta^5)\big\} = \left\{\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right\}.$$

Thus

$$\frac{x(1+x)}{1+x+x^2} = x-x^3+x^4-x^6+x^7-x^9+\cdots = \sum_{n=1}^{\infty}\frac{2}{\sqrt{3}}\,\mathrm{Im}(\eta^{2n-1})x^n.$$

The expression in (2.7) is therefore

$$\begin{aligned}\sum_{n=0}^{k-2}(-1)^n\binom{k-2}{n}\frac{2}{\sqrt{3}}\operatorname{Im}\bigl(\eta^{2(k-1-n)-1}\bigr)\\ =\operatorname{Im}\left(\frac{2\eta}{\sqrt{3}}\sum_{n=0}^{k-2}\binom{k-2}{n}(-1)^n\xi^{k-2-n}\right)=\operatorname{Im}\left(\frac{2\eta}{\sqrt{3}}(\xi-1)^{k-2}\right).\end{aligned}\tag{2.8}$$

Since

$$\xi-1=-\sqrt{3}e^{-\pi i/6},$$

we have

$$\eta(\xi-1)^{k-2}=(-\sqrt{3})^{k-2}e^{(4-k)\pi i/6}.$$

Taking the imaginary part in (2.8) gives (2.5). □

**Lemma 8.** *For every $k\geq 2$ and $i\in\{0,1,2,3\}$,*

$$[t^{2k-i}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t}=3^{k-2}(\xi-1)\xi^{k+i-1},\tag{2.9}$$

*and*

$$[t^{2k-i}]\frac{(1-t)^{k-2}(1-\xi^2 t)^{k-1}}{1-\xi t}=3^{k-2}(1-\xi)\xi^{2k-i}.\tag{2.10}$$

*Proof.* We prove (2.9) by induction on $k$ and leave the analogous verification of (2.10) to the reader. Since

$$1-t^3=(1-t)(1-\xi t)(1-\xi^2 t),$$

Lemma 3 shows that

$$\begin{aligned}\frac{1-\xi t}{1-\xi^2 t}&=\frac{(1-t)(1-\xi t)^2}{1-t^3}\\ &=\frac{1+(\xi^2-\xi)t+(\xi-1)t^2-\xi^2t^3}{1-t^3}\\ &=1+\sum_{j=0}^{\infty}\bigl((\xi^2-\xi)t+(\xi-1)t^2+(1-\xi^2)t^3\bigr)t^{3j}.\end{aligned}\tag{2.11}$$

This is the generating function of an eventually periodic sequence with repeating block

$$\{\xi^2-\xi,\xi-1,1-\xi^2\}.$$

The case $k=2$ follows directly from (2.11).

Assume (2.9) holds for some $k\geq 2$ and all $i\in\{0,1,2,3\}$. Since

$$(1-t)(1-\xi t)=1+\xi^2 t+\xi t^2,$$

we have

$$\frac{(1-t)^{k-1}(1-\xi t)^k}{1-\xi^2 t}=\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t}(1+\xi^2 t+\xi t^2).$$

If $i\geq 2$, let $j=i-2\in\{0,1\}$. By the induction hypothesis,

$$[t^{2k-j}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t}\,\xi^2 t=3^{k-2}(\xi-1)\xi^{k+j+2},\tag{2.12}$$

$$[t^{2k-j}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t}\,\xi t^2=3^{k-2}(\xi-1)\xi^{k+j+2}.\tag{2.13}$$

Together with the contribution of the constant term, these identities give

$$\begin{aligned}[t^{2(k+1)-i}]\frac{(1-t)^{k-1}(1-\xi t)^k}{1-\xi^2 t}&=3^{k-2}(\xi-1)\xi^{k+j-1}(1+\xi^3+\xi^3)\\ &=3^{k-1}(\xi-1)\xi^{k+i}.\end{aligned}$$

Thus (2.9) holds for $i\in\{2,3\}$ at $k+1$.

For $i = 1$,

$$[t^{2k+1}]\frac{(1-t)^{k-1}(1-\xi t)^k}{1-\xi^2 t} = [t^{2k+1}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t} + 3^{k-2}(\xi-1)\xi^{k-1}\xi^2 + 3^{k-2}(\xi-1)\xi^k\xi. \tag{2.14}$$

The rational function

$$\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t} = \frac{(1-t)^{k-1}(1-\xi t)^k}{1-t^3} \tag{2.15}$$

generates an eventually periodic sequence of period 3, whose repeating part begins at the coefficient of $t^{2k-3}$. Therefore,

$$[t^{2k+1}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t} = [t^{2k-2}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t} = 3^{k-2}(\xi-1)\xi^{k+1}.$$

Substitution into (2.14) gives

$$3^{k-2}(\xi-1)\xi^{k+1} + 3^{k-2}(\xi-1)\xi^{k+1} + 3^{k-2}(\xi-1)\xi^{k+1} = 3^{k-1}(\xi-1)\xi^{k+1}.$$

The case $i = 0$ follows in the same way from the induction hypothesis and the period-3 behavior of (2.15). □

Identity (2.9) does not generally hold for $i > 3$. For example, when $k = 3$ and $i = 4$,

$$[t^2]\frac{(1-t)(1-\xi t)^2}{1-\xi^2 t} = 3\xi - 2 \neq 3(\xi - 1).$$

**Lemma 9.** *For each fixed $m \in \mathbb{N}$,*

$$(x\xi - 1)^m - (\xi - x)^m = z_m f_m(x), \tag{2.16}$$

*where $z_m \in \mathbb{C} \setminus \{0\}$ and $f_m(x) \in \mathbb{Q}[x]$.*

*Proof.* The binomial theorem gives

$$\begin{aligned}(x\xi - 1)^m - (\xi - x)^m &= \sum_{i=0}^{m} \binom{m}{i}\big((x\xi)^i(-1)^{m-i} - (-x)^i\xi^{m-i}\big)\\ &= \sum_{i=0}^{m}(-x)^i\binom{m}{i}\big(\xi^i(-1)^m - \xi^{m-i}\big).\end{aligned}$$

Because $1 + \xi + \xi^2 = 0$,

$$(-1)^m - \xi^m = 0 \iff 2 - (-1)^m(\xi^m + \xi^{2m}) = 0 \iff 6 \mid m.$$

Thus

$$\big\{2 - (-1)^m(\xi^m + \xi^{2m})\big\}_{m\geq 0} = \{0, 1, 3, 4, 3, 1, 0, 1, 3, 4, 3, 1, 0, \ldots\}.$$

If $6 \nmid m$, set $z = (-1)^m - \xi^m$. Then

$$\bar{z} = (-1)^m - \xi^{2m}, \qquad z\bar{z} = 2 - (-1)^m(\xi^m + \xi^{2m}) \in \mathbb{Z}_{>0}.$$

For each $i$,

$$\begin{aligned}\xi^i(-1)^m - \xi^{m-i} &= z\,\frac{(\xi^i(-1)^m - \xi^{m-i})\bar{z}}{z\bar{z}}\\ &= z\,\frac{(\xi^i + \bar{\xi}^i) - (-1)^m(\xi^{m-i} + \bar{\xi}^{m-i})}{2 - (-1)^m(\xi^m + \bar{\xi}^m)} =: zq_{m,i},\end{aligned} \tag{2.17}$$

where $q_{m,i} \in \mathbb{Q}$. If $6 \mid m$, then

$$\xi^i(-1)^m - \xi^{m-i} = \xi^i - \bar{\xi}^i = \sqrt{-3}\,q_{6,i}, \qquad q_{6,i} \in \{-1, 0, 1\}.$$

Consequently,

$$(x\xi - 1)^m - (\xi - x)^m = z_m \sum_{i=0}^{m}(-x)^i\binom{m}{i}q_{m,i},$$

where

$$z_m = \begin{cases}(-1)^m - \xi^m, & 6 \nmid m,\\ \sqrt{-3}, & 6 \mid m,\end{cases} \qquad f_m(x) = \sum_{i=0}^{m}(-x)^i\binom{m}{i}q_{m,i} \in \mathbb{Q}[x],$$

and $q_{m,i} = q_{6,i}$ when $6 \mid m$. □

# 3 Linear and quadratic polynomials

## 3.1 Linear polynomials

According to [6, Theorem 4], if $p(t) = a + bt$, then

$$V^{\mu}_{p(t)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b & a \\ a & b \end{pmatrix}^{\mu} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (a+b)^{\mu+1} \\ (b-a)^{\mu+1} \end{pmatrix},$$

where $a + b$ and $b - a$ are the eigenvalues of $V_{p(t)}$. If $V_{p(t)}$ has period $\mu$, equation (1.3) implies

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (a+b)^{\mu+1} \\ (b-a)^{\mu+1} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{pmatrix} (a+b)^{\mu+1} \\ (b-a)^{\mu+1} \end{pmatrix} = \begin{pmatrix} a+b \\ b-a \end{pmatrix}. \tag{3.1}$$

Assuming $ab \neq 0$, there are three cases.

**Case 1.** If $a = b$, then

$$(2a)^{\mu} = 1, \qquad a = b = \frac{1}{2} e^{2\pi i s/\mu}, \qquad s \in \{1, \dots, \mu\}.$$

**Case 2.** If $a = -b$, then

$$(2b)^{\mu} = 1, \qquad -a = b = \frac{1}{2} e^{2\pi i s/\mu}, \qquad s \in \{1, \dots, \mu\}.$$

**Case 3.** If neither eigenvalue is zero, then

$$(a+b)^{\mu} = (b-a)^{\mu} = 1$$

and

$$(a, b) = \left( \frac{e^{2\pi i s/\mu} - e^{2\pi i \ell/\mu}}{2}, \frac{e^{2\pi i s/\mu} + e^{2\pi i \ell/\mu}}{2} \right),$$

where $s, \ell \in \{1, \dots, \mu\}$ and $s \neq \ell$.

Formula (2.2), with $d = 1$, gives

$$S_{[k]} = h_{2(k-1),k} = [t^{2(k-1)}] \frac{t^k (a+bt)^k}{(1-t)(1-t^2)}, \qquad k \geq 1. \tag{3.2}$$

**Proposition 10.** *For $k \geq 2$,*

$$S_{[k]} = \sum_{n=0}^{k-2} \binom{k}{n} \left\lfloor \frac{k-n}{2} \right\rfloor b^n a^{k-n}.$$

*Proof.* Since

$$\frac{1}{(1-t)(1-t^2)} = \sum_{i=1}^{\infty} i(t^{2i-2} + t^{2i-1}) = \sum_{j=0}^{\infty} \left\lfloor 1 + \frac{j}{2} \right\rfloor t^j,$$

the standard convolution property of coefficient extraction gives

$$\begin{aligned} S_{[k]} = [t^{k-2}] \frac{(a+bt)^k}{(1-t)(1-t^2)} &= \sum_{n=0}^{k-2} [t^n](a+bt)^k [t^{k-2-n}] \frac{1}{(1-t)(1-t^2)} \\ &= \sum_{n=0}^{k-2} \binom{k}{n} b^n a^{k-n} \left\lfloor \frac{k-n}{2} \right\rfloor. \end{aligned}$$

□

To determine when these partial sums are eventually periodic, we use Lemma 3 together with the diagonalization rule [9, Section 2.4]:

$$\mathcal{G}\big([t^n] F(t) \varphi(t)^n\big) = \left. \frac{F(w)}{1 - t\varphi'(w)} \right|_{w = t\varphi(w)}, \tag{3.3}$$

where $\mathcal{G}$ denotes the generating-function operator.

**Theorem 11.** *Let $p(t) = a + bt \in \mathbb{C}[t]$ with $ab \neq 0$. Then $V_{p(t)}$ has period $\mu$ and the column partial sums $\{S_{[k]}\}_{k \geq 1}$ are eventually periodic if and only if*

$$-a = b = \frac{1}{2} e^{2\pi i s/\mu}, \qquad s \in \{1, \dots, \mu\}.$$

*Proof.* Suppose $V_{p(t)}$ has period $\mu$. To apply (3.3) to (3.2), set

$$\varphi(t) = a + bt, \qquad F(t) = \frac{(a+bt)^2}{(1-t)(1-t^2)}.$$

Then $w = ta/(1-tb)$, and a direct calculation gives the generating function for $\{S_{[k]}\}_{k\geq 2}$:

$$\frac{F(w)}{1-tb} = \frac{a^2}{(1+(a-b)t)(1-(a+b)t)^2}. \tag{3.4}$$

The numerator is constant. By Lemma 3, the sequence cannot be eventually periodic if the denominator in (3.4) has a repeated noncancelling root. Therefore $a + b = 0$, and we are in Case 2.

Conversely, suppose

$$-a = b = \frac{1}{2}e^{2\pi is/\mu}.$$

Then

$$V_{p(t)} = a\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad V_{p(t)}^{n+1} = (-2a)^n V_{p(t)}.$$

Since $(-2a)^\mu = 1$, the matrix has period $\mu$. Moreover, (3.4) reduces to

$$\frac{a^2}{1+2at} = a^2\big(1 + (-2a)t + (-2a)^2t^2 + (-2a)^3t^3 + \cdots\big),$$

so $\{S_{[k]}\}_{k\geq 1}$ is eventually periodic. Explicitly,

$$S_{[k]} = a^2(-2a)^{k-2}, \qquad k \geq 2. \tag{3.5}$$

□

**Corollary 12.** *For every $k \geq 2$,*

$$\sum_{n=0}^{k}(-1)^n\binom{k}{n}\left\lfloor\frac{k-n}{2}\right\rfloor = (-2)^{k-2}.$$

*Proof.* Take $a = 1$ and $b = -1$ in Proposition 10 and (3.5), and note that

$$\left\lfloor\frac{k-n}{2}\right\rfloor = 0 \qquad \text{when } n \in \{k-1, k\}.$$

□

## 3.2 Quadratic polynomials

Let

$$p(t) = a + bt + ct^2.$$

Then

$$V_{p(t)} = \begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}, \qquad F^{-1}V_{p(t)}F = D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where

$$F = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}, \qquad F^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{pmatrix},$$

and

$$\lambda_1 = a + b + c, \qquad \lambda_2 = a\xi^2 + b\xi + c, \qquad \lambda_3 = a\xi + b\xi^2 + c.$$

If $V_{p(t)}$ has period $\mu$, then $V_{p(t)}^{\mu+1} = V_{p(t)}$, and hence

$$FD^{\mu+1}F^{-1} = V_{p(t)}^{\mu+1} = V_{p(t)} = FDF^{-1}.$$

Thus

$$\begin{cases} (a+b+c)^{\mu+1} = a+b+c, \\ (a\xi^2+b\xi+c)^{\mu+1} = a\xi^2+b\xi+c, \\ (a\xi+b\xi^2+c)^{\mu+1} = a\xi+b\xi^2+c. \end{cases} \tag{3.6}$$

For a quadratic polynomial whose matrix has period $\mu$, consider

$$S_{3(k-1)} = \sum_{i=0}^{3(k-1)} C_{i,k},$$

the $3(k-1)$-st partial sum of

$$\sum_{i=0}^{\infty} C_{i,k}t^i = \frac{\big(t(a+bt+ct^2)\big)^k}{1-t^3}.$$

By [6, Theorems 1 and 2], the periodic part of the $k$-th column begins with the $\big(1+3(k-1)\big)$-st term and equals

$$\begin{pmatrix} a_{k-1,0} \\ a_{k-1,1} \\ a_{k-1,2} \end{pmatrix} = V_{p(t)}^{k-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Using the polynomial

$$p_{k-1}(t) = a_{k-1,0} + a_{k-1,1}t + a_{k-1,2}t^2$$

from Definition 4, the infinite periodic tail of the column has generating function

$$\frac{t^{1+3(k-1)}p_{k-1}(t)}{1-t^3}.$$

Consequently,

$$S_{3(k-1)} = \lim_{t\to 1}\left[\frac{(tp(t))^k}{1-t^3} - \frac{t^{1+3(k-1)}p_{k-1}(t)}{1-t^3}\right]. \tag{3.7}$$

Assume that the partial sums are eventually periodic with period $h \in \mathbb{N}$. For $k \in \{1,\ldots,h\}$ and $n \geq 0$,

$$S_{3(k-1)} = S_{3(k+nh-1)}.$$

Taking $n = \mu$ in (3.7) gives

$$\lim_{t\to 1}\left[\frac{(tp(t))^k}{1-t^3} - \frac{t^{1+3(k-1)}p_{k-1}(t)}{1-t^3}\right] = \lim_{t\to 1}\left[\frac{(tp(t))^{k+h\mu}}{1-t^3} - \frac{t^{1+3(k+h\mu-1)}p_{k+h\mu-1}(t)}{1-t^3}\right]. \tag{3.8}$$

Because $p_{k+h\mu-1} = p_{k-1}$,

$$\frac{t^{1+3(k-1)}p_{k-1}(t)}{1-t^3} - \frac{t^{1+3(k+h\mu-1)}p_{k+h\mu-1}(t)}{1-t^3} = t^{1+3(k-1)}\sum_{j=0}^{h\mu-1} t^{3j}p_{k-1}(t). \tag{3.9}$$

Since $p_{k-1}(1) = (a+b+c)^k$ by Lemma 5, equation (3.8) is equivalent to

$$(a+b+c)^k \lim_{t\to 1}\frac{1-(tp(t))^{h\mu}}{1-t^3} - (a+b+c)^k h\mu = 0. \tag{3.10}$$

We first assume that $\lambda_1 = a+b+c \neq 0$. Then $\lambda_1^{\mu} = 1$, and l'Hôpital's rule gives

$$\lim_{t\to 1}\frac{1-(tp(t))^{h\mu}}{1-t^3} = \frac{h\mu(a+2b+3c)}{3(a+b+c)}.$$

Equation (3.10) therefore implies

$$a+2b+3c = 3a+3b+3c, \qquad \text{so} \qquad b = -2a.$$

Substitution into (3.6) gives

$$\begin{cases} (c-a)^{\mu} = 1, \\ (a\xi^2 - 2a\xi + c)^{\mu+1} = a\xi^2 - 2a\xi + c, \\ (a\xi - 2a\xi^2 + c)^{\mu+1} = a\xi - 2a\xi^2 + c. \end{cases} \tag{3.11}$$

Neither of the latter two eigenvalues can be zero. For example, if $a\xi^2 - 2a\xi + c = 0$, then $(c-a)^{\mu} = (3a\xi)^{\mu} = 1$, while the third equation would force $(1-\xi)^{\mu} = 1$, impossible because $|1-\xi| = \sqrt{3}$. Thus (3.11) is equivalent to

$$\begin{cases} (c-a)^{\mu} = 1, \\ (c-a-3a\xi)^{\mu} = 1, \\ (c-a-3a\xi^2)^{\mu} = 1. \end{cases} \tag{3.12}$$

The three quantities in (3.12) have equal modulus. Applying Lemma 6 to $(c-a)\xi$ and $3a$ shows that $c = ra$ for some $r \in \mathbb{R}$. The equality $|1 - 3\xi/(r-1)| = 1$ has the unique solution $r = -2$. Hence

$$p(t) = a(1 - 2t - 2t^2).$$

System (3.12) becomes

$$(-3a)^\mu = 1, \qquad \big(-3a(1+\xi)\big)^\mu = 1, \qquad \big(-3a(1+\xi^2)\big)^\mu = 1. \tag{3.13}$$

It follows that $(3a)^\mu = 1$ and $6 \mid \mu$. Conversely, these conditions make the matrix periodic, and the argument leading to (3.10) shows that the partial sums have the same period.

**Proposition 13.** *Let $p(t) = a + bt + ct^2$ satisfy $acp(1) \neq 0$. Then $V_{p(t)}$ has finite period $\mu$ and the partial sums $\{S_{3(k-1)}\}_{k\geq 1}$ are eventually periodic if and only if*

$$p(t) = a(1 - 2t - 2t^2), \qquad 6 \mid \mu, \qquad (3a)^\mu = 1.$$

Now suppose that $\lambda_1 = a + b + c = 0$. If also $\lambda_2 = 0$, then $b = a\xi^2$ and $c = a\xi$, so

$$p(t) = a(1 + \xi^2 t + \xi t^2) = a(t-1)(\xi t - 1), \qquad (3a\xi)^\mu = 1.$$

For $k \geq 2$, Lemma 8 gives

$$S_{[k]} = [t^{2k-3}]\frac{\big(a(t-1)(\xi t-1)\big)^k}{(1-t)^2(1+t+t^2)} = a^k[t^{2k-3}]\frac{(1-t)^{k-2}(1-\xi t)^{k-1}}{1-\xi^2 t} = (3a\xi)^k\frac{\xi-1}{9\xi}. \tag{3.14}$$

Similarly, if $\lambda_1 = \lambda_3 = 0$, then

$$p(t) = a(\xi^2 + t + \xi t^2) = a(t-1)(\xi t - \xi^2), \qquad (3a\xi^2)^\mu = 1,$$

and

$$S_{[k]} = (3a\xi^2)^k\frac{1-\xi}{9}, \qquad k \geq 2. \tag{3.15}$$

**Proposition 14.** *Let $p(t) = a + bt + ct^2$ satisfy $ac \neq 0$, $a + b + c = 0$, and suppose exactly one eigenvalue of $V_{p(t)}$ is nonzero. Then $V_{p(t)}$ has finite period $\mu$ and the partial sums are eventually periodic if and only if either*

$$p(t) = a(1 + \xi^2 t + \xi t^2), \quad (3a\xi)^\mu = 1, \quad S_{[k]} = (3a\xi)^k\frac{\xi-1}{9\xi},$$

*or*

$$p(t) = a(\xi^2 + t + \xi t^2), \quad (3a\xi^2)^\mu = 1, \quad S_{[k]} = (3a\xi^2)^k\frac{1-\xi}{9},$$

*where $k \geq 2$.*

It remains to consider $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$. Since $c = -a - b$, the last two equations of (3.6) become

$$\begin{cases} \big((\xi-1)a(r-\xi^2)\big)^\mu = 1, \\ \big((\xi-1)a(1-\xi^2 r)\big)^\mu = 1, \end{cases} \qquad r = \frac{b}{a}, \tag{3.16}$$

whenever $a \neq 0$. Dividing the equations shows that

$$1 = \left(\frac{b - \xi^2 a}{a - \xi^2 b}\right)^\mu = \left(\frac{b/a - \xi^2}{1 - \xi^2 b/a}\right)^\mu, \tag{3.17}$$

and therefore

$$\left|\frac{b}{a} - \xi^2\right| = \left|\frac{b}{a} - \xi\right|. \tag{3.18}$$

If $b = 0$, then $p(t) = a(1-t^2)$, $6 \mid \mu$, and $\big(a(\xi-1)\big)^\mu = 1$. Formula (2.2), followed by Lemma 7, yields

$$S_{[k]} = \frac{2}{3\sqrt{3}}(-a\sqrt{3})^k \sin\left(\frac{(k-4)\pi}{6}\right), \qquad k \geq 1. \tag{3.19}$$

Since $|a\sqrt{3}| = 1$, this sequence is periodic.

There is also a generating-function proof. The calculation in Lemma 7 gives

$$S_{[k]} = [x^{k-2}]\frac{a^2(1+x)(a-ax)^{k-2}}{1+x+x^2}. \tag{3.20}$$

Writing this in the diagonal form

$$\mathcal{G}\left([x^{k-2}]F(x)\varphi(x)^{k-2}\right), \tag{3.21}$$

with

$$\varphi(x) = a - ax, \qquad F(x) = \frac{a^2(1+x)}{1+x+x^2}, \qquad w = \frac{ax}{1+ax},$$

gives

$$\mathcal{G} = \frac{F(w)}{1-x\varphi'(w)} = \frac{a^2(1+2ax)}{1+3ax+3a^2x^2}.$$

Because

$$1+27a^6x^6 = 1-(ia\sqrt{3}x)^6 = (1-3ax+6a^2x^2-9a^3x^3+9a^4x^4)(1+3ax+3a^2x^2),$$

we obtain

$$\mathcal{G} = \frac{a^2(1+2ax)(1-3ax+6a^2x^2-9a^3x^3+9a^4x^4)}{1-(ia\sqrt{3}x)^6} = \frac{Q(x)}{1-(ia\sqrt{3}x)^6} \tag{3.22}$$

for some $Q(x) \in \mathbb{C}[x]$. Lemma 3 again proves eventual periodicity.

Suppose now that $b \neq 0$. By (3.18) and Lemma 6, $r = b/a$ is a nonzero real number. Thus

$$p(t) = a\big(1+rt-(1+r)t^2\big) = a(1-t)\big(1+(r+1)t\big),$$

and the period conditions are

$$\begin{cases} \big((\xi-1)a(r-\xi^2)\big)^\mu = 1, \\ \big((\xi-1)a(1-\xi^2 r)\big)^\mu = 1. \end{cases} \tag{3.23}$$

These imply

$$(\xi r - 1)^\mu = (\xi - r)^\mu.$$

By Lemma 9, $r$ is a root of a polynomial $f_\mu(x) \in \mathbb{Q}[x]$ of degree $\mu$.

To prove the periodicity of the partial sums, note that Lemma 5 gives $p_m(1) = 0$, so

$$p_m(t) = (1-t)\beta_m(t)$$

for some $\beta_m(t) \in \mathbb{C}[t]$. Formula (3.7) then simplifies to

$$S_{3(k-1)} = \left.\frac{t^k(1+(r+1)t)p_{k-1}(t) - t^{2(k-1)}\beta_{k-1}(t)}{1+t+t^2}\right|_{t=1} = -\frac{\beta_{k-1}(1)}{3}. \tag{3.24}$$

Since $V_{p(t)}$ has period $\mu$, $p_{k-1} = p_{k-1+n\mu}$ and hence $\beta_{k-1} = \beta_{k-1+n\mu}$ for every $n \geq 0$. Therefore the partial sums are periodic.

**Proposition 15.** *Let $p(t) = a+bt+ct^2$ satisfy $ac \neq 0$, $a+b+c=0$, and suppose the other two eigenvalues of $V_{p(t)}$ are nonzero. Then $V_{p(t)}$ has finite period $\mu$ and the partial sums are eventually periodic if and only if*

$$p(t) = a\big(1+rt-(1+r)t^2\big) = a(1-t)\big(1+(r+1)t\big), \tag{3.25}$$

*where $a$ and $r$ satisfy (3.23). In this case $r \in \mathbb{R}$ is a root of a degree-$\mu$ polynomial $f_\mu(t)$ with rational coefficients.*

The polynomials from Lemma 9 can be written as

$$f_\mu(t) = \sum_{i=0}^{\mu}(-t)^i\binom{\mu}{i}\frac{(\xi^i+\bar{\xi}^i)-(-1)^\mu(\xi^{\mu-i}+\bar{\xi}^{\mu-i})}{2-(-1)^\mu(\xi^\mu+\bar{\xi}^\mu)}, \qquad 6 \nmid \mu,$$

and

$$f_\mu(t) = \sum_{i=0}^{\mu}(-t)^i\binom{\mu}{i}\frac{\xi^i-\bar{\xi}^i}{\sqrt{-3}}, \qquad 6 \mid \mu.$$

The first few examples are shown below.

| $\mu$ | $f_\mu(t)$ | roots $r$ |
|---|---|---|
| 1 | $1-t$ | $\{1\}$ |
| 2 | $1-t^2$ | $\{-1,1\}$ |
| 3 | $1+\frac{3}{2}t-\frac{3}{2}t^2-t^3$ | $\{-2,-\frac{1}{2},1\}$ |
| 4 | $1+4t-4t^3-t^4$ | $\{-2-\sqrt{3},-1,-2+\sqrt{3},1\}$ |
| 5 | $1+10t+10t^2-10t^3-10t^4-t^5$ | $\{-8.74,-1.46,-0.68,-0.11,1\}$ |
| 6 | $-6t-15t^2+15t^4+6t^5$ | $\{-2,-1,-\frac{1}{2},0,1\}$ |

Multiplying through by the least common multiple of the denominators gives the following integer-coefficient array. It is not a Riordan array, because every sixth row has a zero on the main diagonal.

$$\begin{array}{rrrrrrrrrr}
1 & -1 \\
1 & 0 & -1 \\
2 & 3 & -3 & -2 \\
1 & 4 & 0 & -4 & -1 \\
1 & 10 & 10 & -10 & -10 & -1 \\
0 & 6 & 15 & 0 & -15 & -6 & 0 \\
1 & -7 & -42 & -35 & 35 & 42 & 7 & -1 \\
1 & 0 & -28 & -56 & 0 & 56 & 28 & 0 & -1 \\
2 & 9 & -36 & -168 & -126 & 126 & 168 & 36 & -9 & -2
\end{array}$$

# 4 Polynomials of degree $d \geq 2$

We now generalize the polynomial $a(1-2t-2t^2)$. Fix $d \geq 2$ and consider

$$p(t) = a\left((d-1) - \sum_{i=1}^{d} 2t^i\right), \qquad \big((d+1)a\big)^k = 1 \quad \text{for some } k \in \mathbb{N}. \tag{4.1}$$

**Theorem 16.** *If $a = 1/(d+1)$, then $V_{p(t)}$ is nonsingular and has order $d+1$ when $d$ is odd, and order $2(d+1)$ when $d$ is even.*

*Proof.* For a circulant matrix generated by $\nu = (\nu_0, \ldots, \nu_d)$, the eigenvalues are

$$\lambda_\ell = \nu_0 + \xi^\ell \nu_1 + \cdots + \xi^{\ell d}\nu_d, \qquad \ell \in \{1, \ldots, d+1\},$$

where here $\xi = e^{2\pi i/(d+1)}$. In our case

$$\nu = \frac{1}{d+1}(-2, -2, \ldots, -2, d-1).$$

For $1 \leq \ell \leq d$, the identity $1 + \xi^\ell + \cdots + \xi^{\ell d} = 0$ gives

$$\xi^\ell \lambda_\ell = -\frac{2}{d+1}(\xi^\ell + \cdots + \xi^{\ell d}) + \frac{d-1}{d+1} = 1.$$

Thus

$$\lambda_\ell = e^{2\pi i(d+1-\ell)/(d+1)}, \quad 1 \leq \ell \leq d, \qquad \lambda_{d+1} = -1.$$

If $d+1$ is even, every eigenvalue has order dividing $d+1$; if $d+1$ is odd, the eigenvalue $-1$ requires the exponent $2(d+1)$. □

**Corollary 17.** *If $(d+1)a$ has order $k$, then $V_{p(t)}$ has order*

$$\operatorname{lcm}(k, d+1) \quad \textit{when } d \textit{ is odd}, \qquad \operatorname{lcm}(k, 2(d+1)) \quad \textit{when } d \textit{ is even}.$$

**Proposition 18.** *Let $d \geq 2$, and let $k$ be the least positive integer such that $((d+1)a)^k = 1$. Then the column partial sums of the polynomial (4.1) are eventually periodic with period*

$$\operatorname{lcm}(k, d+1) \quad \textit{if } d \textit{ is odd}, \qquad \operatorname{lcm}(k, 2(d+1)) \quad \textit{if } d \textit{ is even}.$$

*Proof.* The analogue of (3.7) is

$$S_{(k-1)(d+1)} = \lim_{t\to 1}\left[\frac{(tp(t))^k}{1-t^{d+1}} - \frac{t^{1+(k-1)(d+1)}p_{k-1}(t)}{1-t^{d+1}}\right]. \tag{4.2}$$

Set

$$A_k = \frac{(tp(t))^k}{1-t^{d+1}}, \qquad B_k = \frac{t^{1+(k-1)(d+1)}p_{k-1}(t)}{1-t^{d+1}},$$

and let $\mu$ be the period stated in the proposition. It is enough to prove

$$\lim_{t\to 1}(A_k - B_k) - \lim_{t\to 1}(A_{k+h\mu} - B_{k+h\mu}) = 0 \qquad (h \in \mathbb{N}_0). \tag{4.3}$$

Since $p_{k+h\mu-1} = p_{k-1}$,

$$B_{k+h\mu} - B_k = -t^{1+(k-1)(d+1)}p_{k-1}(t)\sum_{j=0}^{h\mu-1} t^{j(d+1)}.$$

Lemma 5 gives

$$p_{k-1}(1) = \left(\sum_{i=0}^{d} a_i\right)^k = \big(-a(d+1)\big)^k = (-1)^k,$$

and hence

$$\lim_{t\to 1}(B_{k+h\mu} - B_k) = (-1)^{k+1}h\mu. \tag{4.4}$$

On the other hand, l'Hôpital's rule, $p(1)^\mu = 1$, and $p'(1) = -ad(d+1)$ give

$$\lim_{t\to 1}(A_k - A_{k+h\mu}) = (-1)^k \lim_{t\to 1}\frac{1-(tp(t))^{h\mu}}{1-t^{d+1}} = (-1)^k\frac{h\mu}{(d+1)p(1)}\big(p(1)+p'(1)\big) = (-1)^k h\mu.$$

This cancels (4.4), proving (4.3). □

**Example 19.** Let $d = 3$ and $k = 5$. The choices

$$a = \frac{1}{4}, \qquad a = \frac{e^{6\pi i/5}}{4}$$

satisfy $((d+1)a)^k = 1$, and

$$p(t) = a(2 - 2t - 2t^2 - 2t^3).$$

For $a = 1/4$, the partial sums have period 4 and repeating block

$$\left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}.$$

For $a = e^{6\pi i/5}/4$, the period is 20. With $\zeta = e^{2\pi i/10}$, one repeating block is

$$\left\{0, -\frac{\zeta^2}{2}, -\frac{\zeta^3}{2}, 0, 0, \frac{\zeta}{2}, \frac{\zeta^2}{2}, 0, 0, -\frac{1}{2}, -\frac{\zeta}{2}, 0, 0, -\frac{\zeta^4}{2}, \frac{1}{2}, 0, 0, \frac{\zeta^6}{2}, \frac{\zeta^8}{2}, 0\right\}.$$

# 5 Graphs of column partial sums

An eventually periodic sequence takes only finitely many values. For a complex-valued sequence, connect two points in the complex plane whenever they represent consecutive terms. In this way every eventually periodic sequence $\{S_{[k]}\}_{k\geq 1}$ produces a finite planar graph. The following graphs were produced with Wolfram Mathematica.

## 5.1 Polynomials $p(t) = a(1-t)$

By Theorem 11, eventual periodicity occurs when

$$a = -\frac{1}{2}e^{2\pi i s/\mu}, \qquad s \in \{1, \ldots, \mu\},$$

and

$$S_{[k]} = a^2(-2a)^{k-2}, \qquad k \geq 2.$$

Except for $S_{[1]} = 0$, all vertices lie on a circle. Conjugate choices of $a$ produce graphs reflected across the real axis, so Figure 1 shows one representative from each conjugate pair for $\mu = 14$.

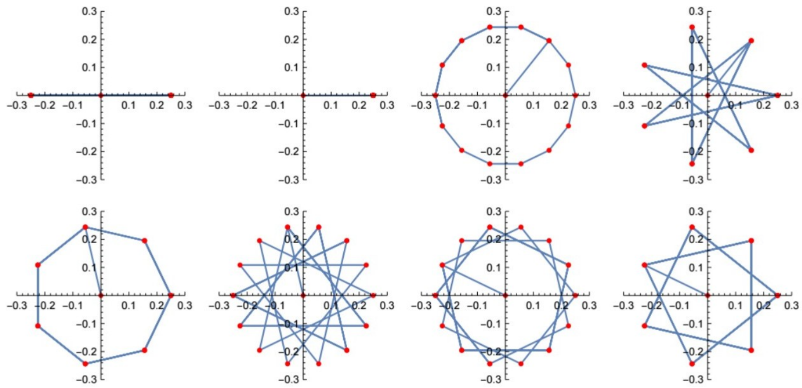

Figure 1: $p(t) = a(1-t)$, where $-2a = e^{2\pi i s/14}$.

## 5.2 Polynomials $a(1-2t-2t^2)$

Here $6 \mid \mu$ and $(3a)^\mu = 1$. For $\mu = 12$, representatives of the conjugate pairs are

$$a \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{i}{3}, \frac{e^{\pi i/6}}{3}, \frac{e^{\pi i/3}}{3}, \frac{e^{2\pi i/3}}{3}, \frac{e^{5\pi i/6}}{3}\right\}.$$

For example, $a = i/3$ gives a sequence of period 12 with repeating block

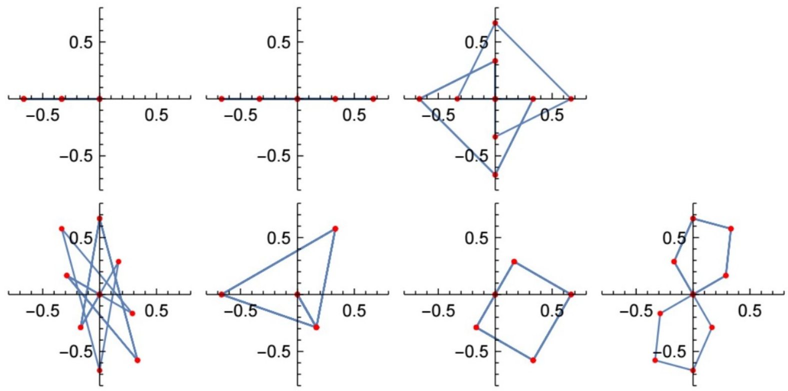

Figure 2: $p(t) = a(1-2t-2t^2)$, where $(3a)^{12} = 1$.

$$\left\{0, \frac{1}{3}, -\frac{2i}{3}, -\frac{2}{3}, \frac{i}{3}, 0, 0, -\frac{1}{3}, \frac{2i}{3}, \frac{2}{3}, -\frac{i}{3}, 0\right\}.$$

## 5.3 The two singular quadratic families

For the polynomials

$$a(1+\xi^2 t+\xi t^2) \quad \text{and} \quad a(\xi^2+t+\xi t^2),$$

the partial sums are respectively

$$S_{[k]} = (3a\xi)^k \frac{\xi-1}{9\xi}, \qquad S_{[k]} = (3a\xi^2)^k \frac{1-\xi}{9}.$$

Again, the nonzero vertices lie on a circle. Figure 3 shows the second family for $\mu = 7$. For $a = \xi/3$, the first four terms are

$$\left\{0, \frac{i}{3\sqrt{3}}, \frac{3-i\sqrt{3}}{18}, \frac{-3-i\sqrt{3}}{18}\right\}.$$

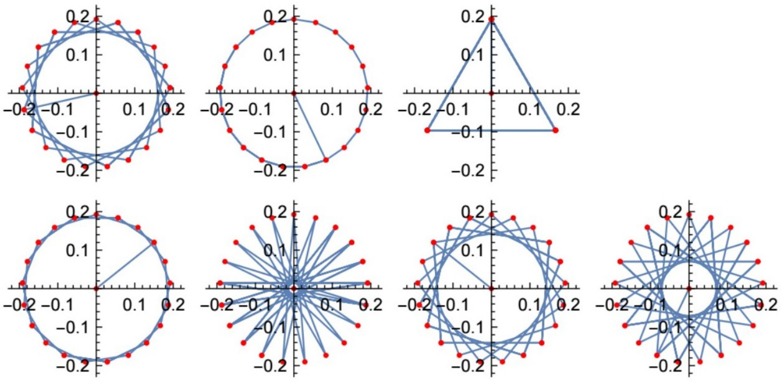


Figure 3: $p(t) = a(\xi^2 + t + \xi t^2)$, where $(3a\xi^2)^7 = 1$.

### 5.4 Polynomials $a(1 + rt - (1 + r)t^2)$, with $r \in \mathbb{R}$

For $r = 0$, formula (3.19) places the points on equally spaced rays from the origin, with their distances controlled by $\sin((k - 4)\pi/6)$. We therefore focus on $r \neq 0$. Conjugate values of $a$ again give reflected graphs. Since the formula for $f_\mu$ changes when $6 \mid \mu$, Figures 4 and 5 illustrate $\mu = 14$ and $\mu = 12$.

For $\mu = 14$, $f_{14}$ has fourteen real roots; its largest is approximately 11.0563, so the factor $1 + (r + 1)t$ is approximately $1 + 12.0563t$. For $\mu = 12$, the eleven roots are

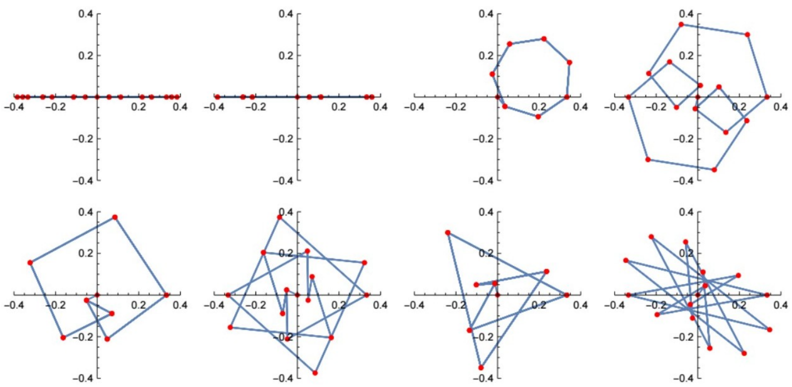


Figure 4: $p(t) = a(1 - t)(1 + 12.0563t)$, with $\mu = 14$.

$$\left\{-2, -1, -\frac{1}{2}, 0, 1, -2 - \sqrt{3}, \frac{-1 - \sqrt{3}}{2}, 1 - \sqrt{3}, -2 + \sqrt{3}, \frac{-1 + \sqrt{3}}{2}, 1 + \sqrt{3}\right\}.$$

The largest is $1 + \sqrt{3} \approx 2.732$, so $1 + (r + 1)t \approx 1 + 3.732t$.

### 5.5 Polynomials $a(2 - 2t - 2t^2 - 2t^3)$

Take one representative from each conjugate pair of solutions of $(4a)^5 = 1$. The three essentially different graphs appear in Figure 6. The first two correspond to $a = 1/4$ and $a = e^{6\pi i/5}/4$, and have periods 4 and 20, respectively, as in Example 19.

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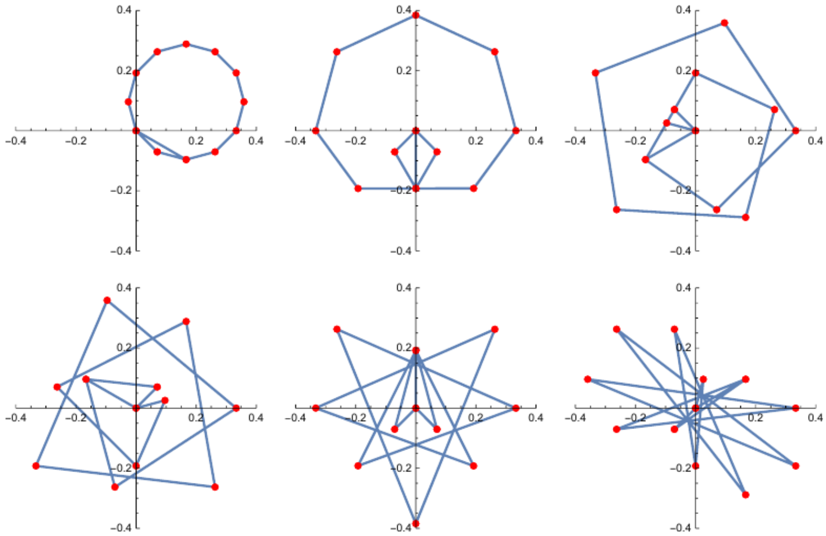

Figure 5: $p(t) = a(1-t)(1+3.732t)$, with $\mu = 12$.

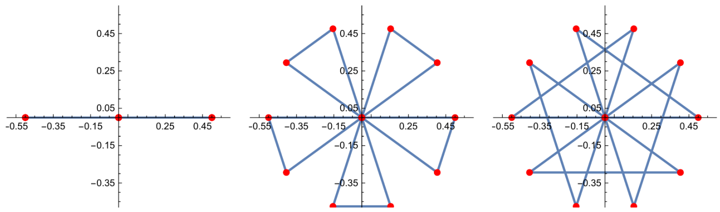

Figure 6: $p(t) = a(2-2t-2t^2-2t^3)$, where $(4a)^5 = 1$.


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