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Finding an Isomorphism Between the Riordan Group and a Subgroup of the Double Riordan Group

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Abstract: The Riordan group $\mathcal{R}$ is a set of infinite lower-triangular matrices defined by two generating functions, g and f. The elements of $\mathcal{R}$ are called Riordan arrays and denoted by (g, f), where f is the multiplier function of the array. The 0^{th} column of a Riordan array is defined by g, and the k^{th} column is given by the generating function gf^k. Similarly, the Double Riordan group $\mathcal{DR}$ is a set of infinite lower-triangular matrices defined by three generating functions, g, f_1, and f_2, where g is an even function and the multiplier functions f_1 and f_2 are odd.

Since the introduction of the Double Riordan group by Davenport, Shapiro, and Woodson in 2011, an open question remained as to whether there is a subgroup of the Double Riordan group that is isomorphic to the Riordan group. This paper introduces monomorphism theorems that answer this question and extend the results to the k-Riordan group. We also use these theorems to define elements of the Double Riordan group that are analogous to pseudo-involutions of the Riordan group.

Keywords: Generating functions; Involutions; Isomorphism; Riordan arrays
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1. Introduction

1.1 The Riordan Group

Let $\mathcal{F}[[z]]$ be the ring of formal power series with coefficients over a field $\mathcal{F}$. The (ordinary) Riordan group $\mathcal{R}$ is a set of infinite lower-triangular matrices that are defined by two (ordinary) generating functions, $g(z)$ and $f(z)$, where $g(z), f(z) \in \mathcal{F}[[z]]$, $g_0 \neq 0$, $f_0 = 0$, and $f_1 \neq 0$. The function $f(z)$ is referred to as the multiplier function, and the elements are called Riordan arrays or Riordan matrices.

The entries of the initial column (indexed as the 0^{th} column) of a Riordan matrix consist of the coefficients (g_n), $n \in \mathbb{N}$, of $g(z)$. The generating function $f(z)$ is multiplied to each successive column such that the matrix has the form

$$(g, f) = \begin{bmatrix} g & gf & gf^2 & gf^3 & \cdots \end{bmatrix}.$$

Formally, the generating functions g and f belong to the groups of formal power series $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. The multiplicative group of units of formal power series is denoted by

$$\mathcal{F}_0 = \{p(z) \in \mathcal{F}[[z]] : p(0) \neq 0\}.$$

The set

$$\mathcal{F}_1 = \{p(z) \in \mathcal{F}[[z]] : p(0) = 0, p'(0) \neq 0\} = z\mathcal{F}_0$$

is a group under composition. Then, for some formal power series $g(z) \in \mathcal{F}_0$ and $f(z) \in \mathcal{F}_1$, the infinite lower-triangular matrix $(d_{n,k})$ with entries

$$d_{n,k} = \begin{cases} [z^n]g(z)f(z)^k, & 0 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases},$$

is a Riordan array.

A common example of a Riordan array is the Pascal matrix,

$$\left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & \cdots & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & & \ddots & & \end{bmatrix}.$$

To determine the product of two Riordan arrays, we first define the product of a Riordan array with an infinite column vector associated with some ordinary generating function $A(z)$. The result is another infinite column vector given by the generating function $B(z)$. This is known as the Fundamental Theorem of Riordan Arrays (FTRA).

Theorem 1. *(The Fundamental Theorem of Riordan Arrays) [6] Let*

$$A(z) = \sum_{k\geq 0} a_k z^k$$

and

$$B(z) = \sum_{k\geq 0} b_k z^k,$$

where $k \in \mathbb{N}$. Also, let A and B be the column vectors $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$. Then $(g, f) * A = B$ if and only if $B(z) = g(z)A(f(z))$.

Proof.

$$\begin{bmatrix} g & gf & gf^2 & gf^3 & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

$$\begin{aligned} a_0 g + a_1 gf + a_2 gf^2 + \cdots &= B(z), \\ g\left(a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \cdots\right) &= B(z), \\ g(z) \cdot (A(z) \circ f(z)) &= B(z). \end{aligned}$$

□

Using the FTRA, we obtain the multiplication rule for Riordan arrays as follows.

Theorem 2. *[6] Let (g, f) and (G, F) be Riordan arrays. Then, the product of the matrices is given by*

$$(g, f) * (G, F) = (gG(f), F(f)).$$

Associativity holds as a result of matrix multiplication. To determine the inverse of a Riordan array, we first provide a theorem on the reciprocal of a formal power series and the definition of a compositional inverse.

Theorem 3. *[7] A formal power series $p(z) = \sum_{k\geq 0} p_k z^k$ has a unique reciprocal formal power series $\frac{1}{p(z)}$ if and only if $p(0) \neq 0$.*

Theorem 4. *[7] Let $p(z) = \sum_{k\geq 0} p_k z^k$ be a formal power series where $p(0) = 0$ and $p'(0) \neq 0$. Then, there exists a unique compositional inverse $\bar{p}(z)$ such that*

$$\bar{p}(z) \circ p(z) = p(z) \circ \bar{p}(z) = z.$$

Let $g(z)$ be an element of $\mathcal{F}_0$. Thus, by Theorem 3, it has a unique reciprocal power series $\frac{1}{g(z)}$. The multiplier function $f(z)$ is an element of $\mathcal{F}_1$, and by Theorem 4 it has a compositional inverse, which we denote by $\bar{f}(z)$. The inverse of a Riordan array (g, f) is given by

$$(g, f)^{-1} = \left(\frac{1}{g(z) \circ \bar{f}(z)}, \bar{f}(z)\right) = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right).$$

Function composition will be expressed as in the last form from now on. The identity element of the Riordan group is the infinite identity matrix, which is denoted by $(1, z)$.

1.2 The Double Riordan Group

Now, let $g(z) \in \mathcal{F}_0$ be an even function and $f_1(z), f_2(z) \in \mathcal{F}_1$ be odd functions. Define the initial column by the generating function $g(z)$ and multiply $f_1(z)$ and $f_2(z)$ alternately to each successive column. Then, the resulting array is called a Double Riordan array. The set of these arrays forms a group called the Double Riordan group. We denote the group by $\mathcal{DR}$ or $2\mathcal{R}$. Double Riordan matrices are denoted by (g, f_1, f_2). The columns of the array are given by

$$(g, f_1, f_2) = \begin{bmatrix} g & gf_1 & g(f_1 f_2) & g(f_1 f_2)f_1 & g(f_1 f_2)^2 & \cdots \end{bmatrix}.$$

To find the product of two Double Riordan arrays, Davenport, Shapiro, and Woodson established the Fundamental Theorem of Double Riordan Arrays (FTDRA) in [3]. When multiplying a $\mathcal{DR}$ array by an infinite column vector $A(z)$, we must consider two cases: when $A(z)$ is an even or odd function. Hence, we have the following theorem.

Theorem 5. *(The Fundamental Theorem of Double Riordan Arrays) Let* $g(z) = \sum_{k\geq 0} g_{2k} z^{2k}$, $f_1(z) = \sum_{k\geq 0} f_{1,2k+1} z^{2k+1}$, *and* $f_2(z) = \sum_{k\geq 0} f_{2,2k+1} z^{2k+1}$.

Case 1: If $A(z) = \sum_{k\geq 0} a_{2k} z^{2k}$ *and* $B(z) = \sum_{k\geq 0} b_{2k} z^{2k}$, *and*

$$A = (a_0, 0, a_2, 0, \cdots)^T \quad \text{and} \quad B = (b_0, 0, b_2, 0, \cdots)^T$$

are column vectors, then

$$(g, f_1, f_2) * A = B \text{ if and only if } B(z) = g(z)A(\sqrt{f_1 f_2}).$$

Case 2: If $A(z) = \sum_{k\geq 0} a_{2k+1} z^{2k+1}$ *and* $B(z) = \sum_{k\geq 0} b_{2k+1} z^{2k+1}$, *then*

$$(g, f_1, f_2) * A = B \text{ if and only if } B(z) = g(z)\sqrt{\frac{f_1}{f_2}}A(\sqrt{f_1 f_2}).$$

They then used the FTDRA to find the product of two Double Riordan Arrays.

Theorem 6. *Let* (g, f_1, f_2) *and* (G, F_1, F_2) *be Double Riordan arrays. Then, the product of the arrays is given by*

$$(g, f_1, f_2) * (G, F_1, F_2) = \left(gG(h), \frac{f_1}{h}F_1(h), \frac{f_2}{h}F_2(h)\right),$$

where $h = \sqrt{f_1 f_2}$.

Note that the product can also be expressed as

$$\left(gG(-h), \frac{f_1}{-h}F_1(-h), \frac{f_2}{-h}F_2(-h)\right)$$

since G is an even function and F_1 and F_2 are odd.

The generating function h is an element of $\mathcal{F}_1$, so there exists a unique compositional inverse that is denoted by $\bar{h}$. Then, for some Double Riordan array (g, f_1, f_2), its inverse is given by

$$(g, f_1, f_2)^{-1} = \left(\frac{1}{g(\bar{h})}, \frac{z\bar{h}}{f_1(\bar{h})}, \frac{z\bar{h}}{f_2(\bar{h})}\right).$$

In [3], Davenport, Shapiro, and Woodson ask whether an isomorphism exists between the Riordan group and a subgroup of the Double Riordan group. In Section 2, we introduce monomorphism theorems that answer this question and extend results to the k-Riordan group. In Section 3, we use these theorems to identify elements of order 2 that are analogous to pseudo-involutions of the Riordan group.

1.3 The k-Riordan Group

Generalizing to matrices with more than two multiplier functions, we define k-Riordan arrays for some positive integer k. Let $g(z) \in \mathcal{F}_0$ and $f_j(z) \in \mathcal{F}_1$, where

$$g(z) = \sum_{n\geq 0} d_{kn} z^{kn}$$

and

$$f_j(z) = \sum_{n\geq 0} f_{j,kn+1} z^{kn+1},$$

$1 \le j \le k$. Then, for $k \ge 1$, the k-Riordan array with respect to g and f_j $(1 \le j \le k)$ is defined by

$$(g, f_1, \ldots, f_k) = \begin{bmatrix} g & gf_1 & gf_1f_2 & \cdots & g\prod_{j=1}^k f_j & gf_1\prod_{j=1}^k f_j & \cdots \end{bmatrix}.$$

The Fundamental Theorem of k-Riordan arrays is established in Theorem 3.6 by T. X. He [4]. The theorem stating the multiplication rule for these arrays follows. For a k-Riordan array $(g, f_1, \ldots, f_k)$, let

$$h(z) = \sqrt[k]{\prod_{j=1}^{k} f_j(z)}.$$

Then, the product of two k-Riordan arrays $(g, f_1, f_2, \ldots, f_k)$ and $(G, F_1, F_2, \ldots, F_k)$ is given by

$$(g, f_1, f_2, \ldots, f_k) * (G, F_1, F_2, \ldots, F_k) = \left(g \cdot G(h), \frac{f_1}{h} \cdot F_1(h), \ldots, \frac{f_k}{h} \cdot F_k(h)\right).$$

1.4 Involutions and Pseudo-involutions

A natural question about an infinite group is which elements have a finite order. When the generating functions $g(z)$ and $f(z)$ of a Riordan array (g, f) are real-valued, i.e. in the set of power series $\mathbb{R}[[z]]$, the matrices with finite order are the identity matrix and those with multiplicative order 2. [1]

Let G be a group with an identity e. Then, $g \in G$ is an involution if and only if $g^2 = e$. Since the Riordan group $\mathcal{R}$ is a group, $R \in \mathcal{R}$ is an involution if and only if $R^2 = (1, z)$. Let the matrix $M = (1, -z)$ and $L \in \mathcal{R}$. When LM or ML is an involution, L is called a pseudo-involution. Note that

$$(LM)^2 = LMLM = I \iff MLM = L^{-1} \iff MLML = I \iff (ML)^2 = I. \quad [2]$$

When a Riordan array is a pseudo-involution, it is conjugate to its inverse. We define $\mathcal{PI}$ to be the set of all pseudo-involutions of the Riordan group. Shapiro and Burstein [1] and Marshall [5] find methods to construct pseudo-involutions, one of which includes Shapiro's Master Theorem and its corollaries in [2].

Suppose $g(z) \in \mathcal{F}_0$. The corresponding $f(z)$ such that (g, f) is a pseudo-involution is referred to as the (pseudo-involutory) companion function of $g(z)$.

Example 1. *Pascal's matrix P is an example of a pseudo-involution in the Riordan group. It has the inverse*

$$P^{-1} = MPM = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & \cdots & \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \\ & & & \ddots & \end{bmatrix}.$$

Note that this inverse is the Pascal matrix with negative signs on alternating diagonals.

Example 2. *The RNA matrix with $g = 1 + zg + z^2g(g-1)$ is also a pseudo-involution, with the array*

$$(g, zg) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 4 & 1 & 0 & 0 & \\ 8 & 13 & 13 & 10 & 5 & 1 & 0 & \\ & & & \vdots & & & & \end{bmatrix}.$$

Solving the functional equation,

$$g(z) = \frac{1 - z \pm \sqrt{4z^4 - 3z^2 - 2z + 1}}{2z^2}.$$

The sequence associated with $g(z)$ (where $\pm$ is set to $-$) can be found in the Online Encyclopedia of Integer Sequences (OEIS) under A098083.

2. The Monomorphism Theorems

2.1 Monomorphisms of $\mathcal{R}$ into $\mathcal{DR}$

Recall that the Double Riordan group $\mathcal{DR}$ is a set of infinite lower-triangular matrices defined by three generating functions $g(z)$, $f_1(z)$ and $f_2(z)$, where $g(z)$ is an even function and the two multiplier functions $f_1(z)$ and $f_2(z)$ are odd. This group was introduced using a construction similar to Riordan arrays. Davenport et al. [3] state that there exists an isomorphic mapping $\psi : \mathcal{R}_C \to \mathcal{DR}$ defined by

$$\psi : (g, f) \mapsto (g, f, f),$$

where $\mathcal{R}_C$ is the Checkerboard subgroup of the Riordan group, $g(z)$ is an even function, and $f(z)$ is an odd function. An open question from [3] asks if there exists an isomorphism between the Riordan group and a subgroup of the Double Riordan group.

Define the map $\mu : \mathcal{R} \to \mathcal{DR}$ by

$$\mu : (g, f) \mapsto \left(g(z^2), \frac{f(z^2)}{z}, \frac{f(z^2)}{z} \right).$$

Although it is clear that μ is a one-to-one function, we show that it fails to be a group homomorphism.

Let $(g, f), (G, F) \in \mathcal{R}$. Then,

$$\begin{aligned}
\mu(g, f) * \mu(G, F) &= \left(g(z^2), \frac{f(z^2)}{z}, \frac{f(z^2)}{z} \right) * \left(G(z^2), \frac{F(z^2)}{z}, \frac{F(z^2)}{z} \right) \\
&= \left(g(z^2) * G\left(\frac{f^2(z^2)}{z^2}\right), \frac{F\left(\frac{f^2(z^2)}{z^2}\right)}{\frac{f(z^2)}{z}}, \frac{F\left(\frac{f^2(z^2)}{z^2}\right)}{\frac{f(z^2)}{z}} \right) \\
&= \left(g(z^2) * G\left(\frac{f^2(z^2)}{z^2}\right), \frac{zF\left(\frac{f^2(z^2)}{z^2}\right)}{f(z^2)}, \frac{zF\left(\frac{f^2(z^2)}{z^2}\right)}{f(z^2)} \right).
\end{aligned}$$

However,

$$\mu((g, f) * (G, F)) = \mu(gG(f), F(f)) = \left(g(z^2)G(f(z^2)), \frac{F(f(z^2))}{z}, \frac{F(f(z^2))}{z} \right).$$

Instead, let us define the map $\phi : \mathcal{R} \to \mathcal{DR}$ by

$$\phi(g, f) = \left(g(z^2), z, \frac{f(z^2)}{z} \right).$$

Theorem 7. *(Monomorphism Theorem I) The map $\phi : \mathcal{R} \to \mathcal{DR}$ defined by*

$$\phi(g, f) = \left(g(z^2), z, \frac{f(z^2)}{z} \right)$$

is a group monomorphism.

Proof. Let (g, f) and (G, F) be two Riordan arrays. To show that the map ϕ is a homomorphism, we prove that

$$\phi(g, f) \cdot \phi(G, F) = \phi((g, f) * (G, F)). \tag{2.1}$$

Note that, for any Riordan array (g, f), the generating functions $g(z) \in \mathcal{F}_0$ and $f(z) \in \mathcal{F}_1$. Thus, the functions $g(z^2)$ and $f(z^2)$ are even. Then, the left-hand side of Equation (2.1) is given by

$$\begin{aligned}
\phi(g, f) \cdot \phi(G, F) &= \left(g(z^2), z, \frac{f(z^2)}{z} \right) * \left(G(z^2), z, \frac{F(z^2)}{z} \right) \\
&= \left(g(z^2)G\left(\left(\sqrt{f(z^2)}\right)^2\right), \sqrt{\frac{z^2}{f(z^2)}}\sqrt{f(z^2)}, \sqrt{\frac{f(z^2)}{z^2}} \frac{F\left(\left(\sqrt{f(z^2)}\right)^2\right)}{\sqrt{f(z^2)}} \right) \\
&= \left(g(z^2)G(f(z^2)), z, \frac{F(f(z^2))}{z} \right).
\end{aligned}$$

The right-hand side of Equation (2.1) is given by

$$\phi((g,f)*(G,F))=\phi(gG(f),F(f))=\left(g(z^2)G(f(z^2)),z,\frac{F(f(z^2))}{z}\right).$$

Thus, ϕ is a homomorphism.

To prove the claim that ϕ is a monomorphism, we show that ϕ is also one-to-one. Assume that, for some Riordan arrays (g,f) and (G,F), we have the mapping $\phi(g,f)=\phi(G,F)$. Then,

$$\left(g(z^2),z,\frac{f(z^2)}{z}\right)=\left(G(z^2),z,\frac{F(z^2)}{z}\right).$$

Clearly, this implies that the functions $g(z^2)=G(z^2)$ and $f(z^2)=F(z^2)$. Therefore, $(g,f)=(G,F)$. $\square$

Corollary 1. *The image of ϕ forms the subgroup*

$$\begin{aligned}\mathcal{A}_1&=\left\{\left(g(z^2),z,\frac{f(z^2)}{z}\right):g(z)\in\mathcal{F}_0 \textit{ and } f(z)\in\mathcal{F}_1\right\}\\&=\{(g,z,f_2):g(z)\in\mathcal{F}_0 \textit{ is an even function and } f_2(z)\in\mathcal{F}_1 \textit{ is an odd function}\}.\end{aligned}$$

is a subgroup of the Double Riordan group.

Since z is one of the multiplier functions, we define the subgroup $\mathcal{A}_1$ to be the **type-1 almost Appell** subgroup of the Double Riordan group.

Corollary 2. *The Riordan group is isomorphic to the type-1 almost Appell subgroup of the Double Riordan group.*

Proof. Since ϕ is a group monomorphism, $\mathrm{Ker}(\phi)$ is the trivial subgroup of the Riordan group. By the First Isomorphism Theorem,

$$\mathcal{A}_1=\mathrm{Im}(\phi)\cong\mathcal{R}/\mathrm{Ker}(\phi)\cong\mathcal{R}.$$

$\square$

The function ϕ can be viewed as a transformation of matrices as follows. Denote the n^{th} column of any matrix A by $[Col_n(A)]$, and associate the column with its generating function. Then,

$$[Col_{2n}(\phi(g,f))]=g(z^2)\left(\frac{f(z^2)}{z}\right)^n$$

and

$$[Col_{2n+1}(\phi(g,f))]=zg(z^2)\left(\frac{f(z^2)}{z}\right)^n.$$

It is not difficult to show that the map $\psi:\mathcal{R}\to\mathcal{DR}$ defined by

$$\psi(g,f)=\left(g(z^2),\frac{f(z^2)}{z},z\right)$$

is also a group monomorphism. We define the set

$$\begin{aligned}\mathcal{A}_2&=\left\{\left(g(z^2),\frac{f(z^2)}{z},z\right):g(z)\in\mathcal{F}_0 \text{ and } f(z)\in\mathcal{F}_1\right\}\\&=\{(g,f_1,z):g(z)\in\mathcal{F}_0 \text{ is an even function and } f_1(z)\in\mathcal{F}_1 \text{ is an odd function}\}\end{aligned}$$

to be the type-2 almost Appell subgroup of the Double Riordan group. We note that the intersection $\mathcal{A}_1\cap\mathcal{A}_2=\mathcal{A}$, where $\mathcal{A}$ is the Appell subgroup of the Double Riordan group. Therefore, we know that at least two copies of the Riordan group are found in the Double Riordan group.

2.2 Monomorphisms into Higher Riordan Groups

Let k be a positive integer. Extending this mapping to the k-Riordan group for $k>2$, we get the following corollary. We define the set

$$S_k=\left\{\left(g(z^k),\underbrace{z,z,\ldots,z}_{k-1},\frac{f(z^k)}{z^{k-1}}\right):g(z)\in\mathcal{F}_0 \text{ and } f(z)\in\mathcal{F}_1\right\}$$

and show that the Riordan group is isomorphic to S_k.

Corollary 3. *Denoting the k-Riordan group by $k\mathcal{R}$, the map $\phi_k : \mathcal{R} \to k\mathcal{R}$ defined by*

$$\phi_k(g,f) = \left(g(z^k), \underbrace{z, z, \ldots, z}_{k-1}, \frac{f(z^k)}{z^{k-1}} \right)$$

is a group monomorphism. Also, the set S_k is a subgroup of the k-Riordan group. Furthermore, the map $\phi_k : \mathcal{R} \to k\mathcal{R}$ is a group monomorphism for any permutation of the k multiplier functions $z, z, \ldots, z, \frac{f(z^k)}{z^{k-1}}$, and the Riordan group has k known isomorphic copies in the k-Riordan group.

Proof. Let (g, f) and (G, F) be Riordan arrays, and let $h = \sqrt[k]{f(z^k)}$. Note that this implies $h^k = f(z^k)$. Then,

$$\begin{aligned}
\phi_k(g,f) * \phi_k(G,F) &= \left(g(z^k), \underbrace{z, z, \ldots, z}_{k-1}, \frac{f(z^k)}{z^{k-1}} \right) * \left(G(z^k), \underbrace{z, z, \ldots, z}_{k-1}, \frac{F(z^k)}{z^{k-1}} \right) \\
&= \left(g(z^k)G(f(z^k)), \underbrace{\frac{z}{h}h, \ldots, \frac{z}{h}h}_{k-1}, \frac{\frac{f(z^k)}{z^{k-1}}}{h} \frac{F(f(z^k))}{h^{k-1}} \right) \\
&= \left(g(z^k)G(f(z^k)), \underbrace{z, \ldots, z}_{k-1}, \frac{f(z^k)}{z^{k-1}} \frac{F(f(z^k))}{h^k} \right) \\
&= \left(g(z^k)G(f(z^k)), \underbrace{z, \ldots, z}_{k-1}, \frac{F(f(z^k))}{z^{k-1}} \right).
\end{aligned}$$

Also,

$$\phi_k((g,f) * (G,F)) = \phi_k(gG(f), F(f)) = \left(g(z^k)G(f(z^k)), \underbrace{z, z, \ldots, z}_{k-1}, \frac{F(f(z^k))}{z^{k-1}} \right).$$

Thus, ϕ_k is a group homomorphism. It is straightforward to see that the homomorphism is also one-to-one. Furthermore, it is clear that the map from $\mathcal{R}$ to S_k is onto. Thus, S_k is a subgroup of the k-Riordan group.

A similar proof shows that rearranging the multiplier functions of elements in S_k and mapping the Riordan group to that set is also a group monomorphism. □

A natural question is, for $k \geq 3$, do there exist isomorphic copies of the $(k-1)$-Riordan group in the k-Riordan group? If so, how many copies can we establish? We answer these questions in the following theorems.

Theorem 8. *(Monomorphism Theorem II) Let $g(z) \in \mathcal{F}_0$ and $f_1(z), f_2(z) \in \mathcal{F}_1$. Then,*

$$\mathcal{DR} = \left\{ \left(g(z^2), \frac{f_1(z^2)}{z}, \frac{f_2(z^2)}{z} \right) \right\}. \tag{2.2}$$

Let the subset

$$S = \left\{ \left(g(z^3), z, \frac{f_1(z^3)}{z^2}, \frac{f_2(z^3)}{z^2} \right) : g(z) \in \mathcal{F}_0 \text{ and } f_1(z), f_2(z) \in \mathcal{F}_1 \right\}.$$

The map $\chi : \mathcal{DR} \to 3\mathcal{R}$ defined by

$$\chi \left(g(z^2), \frac{f_1(z^2)}{z}, \frac{f_2(z^2)}{z} \right) = \left(g(z^3), z, \frac{f_1(z^3)}{z^2}, \frac{f_2(z^3)}{z^2} \right)$$

is a group monomorphism. Furthermore, S is a subgroup of the Triple Riordan group and is isomorphic to the Double Riordan group. There are three known isomorphic copies of the Double Riordan group in the Triple Riordan group.

Proof. Select generating functions $g(z), G(z) \in \mathcal{F}_0$ and $f_1(z), f_2(z), F_1(z), F_2(z) \in \mathcal{F}_1$. Then, the matrices

$$D_1 = \left(g(z^2), \frac{f_1(z^2)}{z}, \frac{f_2(z^2)}{z} \right) \quad \text{and} \quad D_2 = \left(G(z^2), \frac{F_1(z^2)}{z}, \frac{F_2(z^2)}{z} \right)$$

are Double Riordan arrays. Also, let
$$h(z^3) = \sqrt[3]{\frac{f_1(z^3)f_2(z^3)}{z^3}}$$
and note that
$$h^3(z^3) = \frac{f_1(z^3)f_2(z^3)}{z^3}.$$
To show that χ is a group homomorphism, we must show that $\chi(D_1) \cdot \chi(D_2) = \chi(D_1 * D_2)$.

The left-hand side of the equation is given by

$$\begin{aligned}
\chi(D_1) \cdot \chi(D_2) &= \chi\left(g(z^2), \frac{f_1(z^2)}{z}, \frac{f_2(z^2)}{z}\right) \cdot \chi\left(G(z^2), \frac{F_1(z^2)}{z}, \frac{F_2(z^2)}{z}\right) \\
&= \left(g(z^3), z, \frac{f_1(z^3)}{z^2}, \frac{f_2(z^3)}{z^2}\right) \cdot \left(G(z^3), z, \frac{F_1(z^3)}{z^2}, \frac{F_2(z^3)}{z^2}\right) \\
&= \left(g(z^3)G(h^3(z^3)), \frac{zh(z^3)}{h(z^3)}, \frac{\frac{f_1(z^3)}{z^2}h(z^3)}{h(z^3)}\frac{F_1(h^3(z^3))}{h^2(z^3)}, \frac{\frac{f_2(z^3)}{z^2}h(z^3)}{h(z^3)}\frac{F_2(h^3(z^3))}{h^2(z^3)}\right) \\
&= \left(g(z^3)G(h^3(z^3)), z, \frac{\frac{f_1(z^3)}{z^2}}{h^3(z^3)}F_1(h^3(z^3)), \frac{\frac{f_2(z^3)}{z^2}}{h^3(z^3)}F_2(h^3(z^3))\right) \\
&= \left(g(z^3)G(h^3(z^3)), z, \frac{f_1(z^3)}{z^2}\frac{z^3}{f_1(z^3)f_2(z^3)}F_1(h^3(z^3)), \frac{f_2(z^3)}{z^2}\frac{z^3}{f_1(z^3)f_2(z^3)}F_2(h^3(z^3))\right) \\
&= \left(g(z^3)G(h^3(z^3)), z, \frac{zF_1(h^3(z^3))}{f_2(z^3)}, \frac{zF_2(h^3(z^3))}{f_1(z^3)}\right).
\end{aligned}$$

We compare this to the right-hand side of the equation. Let
$$p(z^2) = \sqrt{\frac{f_1(z^2)f_2(z^2)}{z^2}}.$$
Then,
$$\begin{aligned}
\chi(D_1 * D_2) &= \chi\left(\left(g(z^2), z, \frac{f_1(z^2)}{z}, \frac{f_2(z^2)}{z}\right) * \left(G(z^2), z, \frac{F_1(z^2)}{z}, \frac{F_2(z^2)}{z}\right)\right) \\
&= \chi\left(g(z^2)G(p^2(z^2)), \frac{zF_1(p^2(z^2))}{f_2(z^2)}, \frac{zF_2(p^2(z^2))}{f_1(z^2)}\right) \\
&= \chi\left(g(z^2)G(p^2(z^2)), \frac{1}{z}\frac{z^2F_1(p^2(z^2))}{f_2(z^2)}, \frac{1}{z}\frac{z^2F_2(p^2(z^2))}{f_1(z^2)}\right) \\
&= \left(g(z^3)G(h^3(z^3)), z, \frac{1}{z^2}\frac{z^3F_1(h^3(z^3))}{f_2(z^3)}, \frac{1}{z^2}\frac{z^3F_2(h^3(z^3))}{f_1(z^3)}\right) \\
&= \left(g(z^3)G(h^3(z^3)), z, \frac{zF_1(h^3(z^3))}{f_2(z^3)}, \frac{zF_2(h^3(z^3))}{f_1(z^3)}\right).
\end{aligned}$$

Therefore, χ is a group homomorphism. It is straightforward to verify that the map is one-to-one. Also, it is clear that the Double Riordan group is isomorphic to S, and thus S forms a subgroup of the Triple Riordan group. Similar arguments can be used to prove that at least three isomorphic copies of the Double Riordan group are in the Triple Riordan group.

So far, we have shown that at least two isomorphic copies of the Riordan group are in the Double Riordan group, and at least three isomorphic copies of the Double Riordan group are in the Triple Riordan group. Generalizing these results, we get the theorem below. □

Theorem 9. *(Generalized Monomorphism Theorem) Let k be a positive integer. Then, there are at least $k+1$ copies of the k-Riordan group in the $(k+1)$-Riordan group.*

Proof. Let $g(z) \in \mathcal{F}_0$ and $f_1(z), \ldots, f_k(z) \in \mathcal{F}_1$. Then,
$$k\mathcal{R} = \left\{\left(g(z^k), \frac{f_1(z^k)}{z^{k-1}}, \ldots, \frac{f_k(z^k)}{z^{k-1}}\right)\right\}.$$

Define the maps $\chi_i : k\mathcal{R} \to (k+1)\mathcal{R}$ for integers $1 \le i \le k+1$, such that

$$\begin{cases} \chi_1\left(g(z^k), \dfrac{f_1(z^k)}{z^{k-1}}, \dots, \dfrac{f_k(z^k)}{z^{k-1}}\right) = \left(g(z^{k+1}), z, \dfrac{f_1(z^{k+1})}{z^k}, \dots, \dfrac{f_k(z^{k+1})}{z^k}\right), \\ \chi_2\left(g(z^k), \dfrac{f_1(z^k)}{z^{k-1}}, \dots, \dfrac{f_k(z^k)}{z^{k-1}}\right) = \left(g(z^{k+1}), \dfrac{f_1(z^{k+1})}{z^k}, z, \dots, \dfrac{f_k(z^{k+1})}{z^k}\right), \\ \qquad\qquad\qquad\qquad \vdots \\ \chi_{k+1}\left(g(z^k), \dfrac{f_1(z^k)}{z^{k-1}}, \dots, \dfrac{f_k(z^k)}{z^{k-1}}\right) = \left(g(z^{k+1}), \dfrac{f_1(z^{k+1})}{z^k}, \dots, \dfrac{f_k(z^{k+1})}{z^k}, z\right), \end{cases}$$

where χ_i inserts z as the ith multiplier function in a $(k+1)$-Riordan array. We prove that χ_1 is a group monomorphism and note that the proofs for $\chi_2, \dots, \chi_{k+1}$ follow from a similar argument.

Let

$$\left(g(z^k), \frac{f_1(z^k)}{z^{k-1}}, \dots, \frac{f_k(z^k)}{z^{k-1}}\right) \quad \text{and} \quad \left(G(z^k), \frac{F_1(z^k)}{z^{k-1}}, \dots, \frac{F_k(z^k)}{z^{k-1}}\right)$$

be k-Riordan arrays. Then,

$$\begin{aligned} &\chi_1\left(g(z^k), \frac{f_1(z^k)}{z^{k-1}}, \dots, \frac{f_k(z^k)}{z^{k-1}}\right) * \chi_1\left(G(z^k), \frac{F_1(z^k)}{z^{k-1}}, \dots, \frac{F_k(z^k)}{z^{k-1}}\right) \\ &= \left(g(z^{k+1}), z, \frac{f_1(z^{k+1})}{z^k}, \dots, \frac{f_k(z^{k+1})}{z^k}\right) * \left(G(z^{k+1}), z, \frac{F_1(z^{k+1})}{z^k}, \dots, \frac{F_k(z^{k+1})}{z^k}\right). \end{aligned}$$

Now, let

$$h(z^{k+1}) = \sqrt[k+1]{\frac{\prod_{j=1}^k f_j(z^{k+1})}{z^{(k+1)(k-1)}}},$$

which implies that

$$h^{k+1}(z^{k+1}) = \frac{\prod_{j=1}^k f_j(z^{k+1})}{z^{(k+1)(k-1)}}$$

for the $(k+1)$-Riordan array

$$\left(g(z^{k+1}), z, \frac{f_1(z^{k+1})}{z^k}, \dots, \frac{f_k(z^{k+1})}{z^k}\right).$$

Then,

$$\begin{aligned} &\chi_1\left(g(z^k), \frac{f_1(z^k)}{z^{k-1}}, \dots, \frac{f_k(z^k)}{z^{k-1}}\right) * \chi_1\left(G(z^k), \frac{F_1(z^k)}{z^{k-1}}, \dots, \frac{F_k(z^k)}{z^{k-1}}\right) \\ &= \left(g(z^{k+1}), z, \frac{f_1(z^{k+1})}{z^k}, \dots, \frac{f_k(z^{k+1})}{z^k}\right) * \left(G(z^{k+1}), z, \frac{F_1(z^{k+1})}{z^k}, \dots, \frac{F_k(z^{k+1})}{z^k}\right) \\ &= \left(g(z^{k+1})G(h^{k+1}(z^{k+1})), z, \frac{f_1(z^{k+1})}{z^k}\frac{F_1(h^{k+1}(z^{k+1}))}{h^{k+1}(z^{k+1})}, \dots, \frac{f_k(z^{k+1})}{z^k}\frac{F_k(h^{k+1}(z^{k+1}))}{h^{k+1}(z^{k+1})}\right). \end{aligned}$$

In comparison, let

$$j(z^k) = \sqrt[k]{\frac{\prod_{i=1}^k f_i(z^k)}{z^{k(k-1)}}},$$

which implies that

$$j^k(z^{k+1}) = \frac{\prod_{i=1}^k f_i(z^{k+1})}{z^{(k+1)(k-1)}} = h^{k+1}(z^{k+1}).$$

Then,

$$\begin{aligned} &\chi_1\left(\left(g(z^k), \frac{f_1(z^k)}{z^{k-1}}, \dots, \frac{f_k(z^k)}{z^{k-1}}\right) * \left(G(z^k), \frac{F_1(z^k)}{z^{k-1}}, \dots, \frac{F_k(z^k)}{z^{k-1}}\right)\right) \\ &= \chi_1\left(g(z^k)G(j^k(z^k)), \frac{\frac{f_1(z^k)}{z^{k-1}}}{j(z^k)}\frac{F_1(j^k(z^k))}{j^{k-1}(z^k)}, \dots, \frac{\frac{f_k(z^k)}{z^{k-1}}}{j(z^k)}\frac{F_k(j^k(z^k))}{j^{k-1}(z^k)}\right) \\ &= \chi_1\left(g(z^k)G(j^k(z^k)), \frac{f_1(z^k)}{z^{k-1}}\frac{F_1(j^k(z^k))}{j^k(z^k)}, \dots, \frac{f_k(z^k)}{z^{k-1}}\frac{F_k(j^k(z^k))}{j^k(z^k)}\right) \end{aligned}$$

$$= \left(g(z^{k+1}) G(j^k(z^{k+1})), z, \frac{f_1(z^{k+1})}{z^k} \frac{F_1(j^k(z^{k+1}))}{j^k(z^{k+1})}, \ldots, \frac{f_k(z^{k+1})}{z^k} \frac{F_k(j^k(z^{k+1}))}{j^k(z^{k+1})} \right)$$
$$= \left(g(z^{k+1}) G(h^{k+1}(z^{k+1})), z, \frac{f_1(z^{k+1})}{z^k} \frac{F_1(h^{k+1}(z^{k+1}))}{h^{k+1}(z^{k+1})}, \ldots, \frac{f_k(z^{k+1})}{z^k} \frac{F_k(h^{k+1}(z^{k+1}))}{h^{k+1}(z^{k+1})} \right).$$

Therefore, χ_1 is a group homomorphism. It is straightforward to show that the map is also one-to-one. Also, the set

$$\left\{ \left(g(z^{k+1}), z, \frac{f_1(z^{k+1})}{z^k}, \ldots, \frac{f_k(z^{k+1})}{z^k} \right) : g(z) \in \mathcal{F}_0 \text{ and } f_j(z) \in \mathcal{F}_1, \quad 1 \le j \le k \right\}$$

is a subgroup of the $(k+1)$-Riordan group. □

3. Mono-involutions and Pseudo-involutions

Recall that a Riordan array is an involution if and only if its order is 2. A pseudo-involution R of the Riordan group is defined to have pseudo-order 2 when, given the matrix $M = (1, -z)$, RM or MR is an involution.

Remark 1. *An involution (g, f_1, f_2) of the Double Riordan group satisfies the equation*

$$(g, f_1, f_2)^2 = \left(gg(h), \frac{f_1}{h} f_1(h), \frac{f_2}{h} f_2(h) \right) = (1, z, z),$$

where $h = \sqrt{f_1 f_2}$.

To define mono-involutions in the Double Riordan group, we refer to the monomorphisms from Section 2. These maps send involutions in the Riordan group to involutions of the Double Riordan group.

Definition 1. Define $\phi : \mathcal{R} \to \mathcal{DR}$ to be the monomorphism

$$\phi(g, f) = \left(g(z^2), z, \frac{f(z^2)}{z} \right),$$

where $g \in \mathcal{F}_0$ and $f \in \mathcal{F}_1$. Similarly, define $\psi : \mathcal{R} \to \mathcal{DR}$ to be the monomorphism

$$\psi(g, f) = \left(g(z^2), \frac{f(z^2)}{z}, z \right).$$

Let $M = (1, -z)$. Then, we denote the matrix $\phi(M) = (1, z, -z)$ by M_1 and the matrix $\psi(M) = (1, -z, z)$ by M_2.

- A Double Riordan array D is called a type-1 mono-involution if DM_1 or $M_1 D$ is an involution.
- When DM_2 or $M_2 D$ is an involution, we call the array a type-2 mono-involution.
- We define a pseudo-involution of the Double Riordan group to be an element with pseudo-order 2 for $M_3 = (1, -z, -z)$.

We note that there is no known monomorphism $\Delta : \mathcal{R} \to \mathcal{DR}$ such that the matrix M_3 is an image of a Riordan array.

We state the following lemma, which is proved by direct computation.

Lemma 1. *Let $M_1, M_2, M_3 \in \mathcal{DR}$ be given by*

$$M_1 = (1, z, -z), \qquad M_2 = (1, -z, z), \qquad M_3 = (1, -z, -z).$$

Then,

- $M_1^2 = M_2^2 = M_3^2 = (1, z, z)$.
- $M_1 M_2 = M_2 M_1 = M_3; \qquad M_1 M_3 = M_3 M_1 = M_2; \qquad M_2 M_3 = M_3 M_2 = M_1$.

Theorem 10. *A Double Riordan array (g, f_1, f_2) is a type-1 mono-involution if and only if it is a type-2 mono-involution.*

Proof. Assume that (g, f_1, f_2) is a type-1 mono-involution. Then, for $h = \sqrt{-f_1 f_2}$,

$$\begin{aligned}[(g, f_1, f_2) * M_1]^2 = (g, f_1, -f_2)^2 &= \left(gg(h), \frac{f_1 f_1(h)}{h}, \frac{(-f_2)(-f_2(h))}{h}\right) = (1, z, z) \\ &= \left(gg(h), \frac{(-f_1)(-f_1(h))}{h}, \frac{f_2 f_2(h)}{h}\right) = (g, -f_1, f_2)^2 = [(g, f_1, f_2) * M_2]^2.\end{aligned}$$

□

Thus, from now on, we will simply call such arrays **mono-involutions**.

Theorem 11. *Let $D = (g, f_1, f_2) \in \mathcal{DR}$.*

1. *If (g, f_1, f_2) is a mono-involution, then (g, f_2, f_1) is a mono-involution.*
2. *Let (g, f) be a pseudo-involution of the Riordan group. Then, the images $\phi(g, f)$ and $\psi(g, f)$ are mono-involutions of the Double Riordan group.*
3. *If $D = (g, f_1, f_2)$ is a mono-involution, the arrays $(g, -f_1, f_2)$ and $(g, f_1, -f_2)$ are pseudo-involutions.*

Proof. To prove the first claim, assume that (g, f_1, f_2) is a mono-involution. By the definition of a type-1 mono-involution,

$$[(g, f_1, f_2) * (1, z, -z)]^2 = (1, z, z).$$

Let the generating function $h = \sqrt{-f_1 f_2}$. Then,

$$(1, z, z) = [(g, f_1, f_2) * (1, z, -z)]^2 = (g, f_1, -f_2)^2 = \left(gg(h), \frac{f_1}{h} f_1(h), \frac{-f_2}{h} \cdot (-f_2(h))\right).$$

Since $(g, f_1, -f_2)^2$ is equal to the identity matrix $(1, z, z)$,

$$\frac{f_1}{h} f_1(h) = z = \frac{-f_2}{h} \cdot (-f_2(h)).$$

Thus,

$$\begin{aligned}\left(gg(h), \frac{f_1}{h} f_1(h), \frac{-f_2}{h} \cdot (-f_2(h))\right) &= \left(gg(h), \frac{-f_2}{h} \cdot (-f_2(h)), \frac{f_1}{h} f_1(h)\right) \\ &= (g, -f_2, f_1)^2 = [(g, f_2, f_1) * (1, -z, z)]^2.\end{aligned}$$

Therefore, (g, f_2, f_1) is a type-2 mono-involution. Combining this result with Theorem 10, we prove the first claim.

To prove the second claim, recall the images under the monomorphisms ϕ and ψ. Then, for a pseudo-involution (g, f),

$$\begin{aligned}[\phi(g, f) * M_1]^2 &= [\phi(g, f) * \phi(M)]^2 \\ &= \phi([(g, f) * M]^2) \\ &= \phi(1, z) = (1, z, z).\end{aligned}$$

A similar argument proves that $\psi(g, f)$ is a mono-involution.

To prove the final claim, for a mono-involution $D = (g, f_1, f_2)$,

$$((g, -f_1, f_2) * M_3)^2 = (D * M_2 * M_3)^2 = (DM_1)^2 = (1, z, z).$$

□

The proof of Theorem 11(1) motivates us to look at the mapping $\sigma : \mathcal{DR} \to \mathcal{DR}$ defined in the theorem below.

Theorem 12. *Define $\sigma : \mathcal{DR} \to \mathcal{DR}$ by*

$$\sigma(g, f_1, f_2) = (g, f_2, f_1)$$

for $(g, f_1, f_2) \in \mathcal{DR}$. Then, $\sigma \in \mathrm{Aut}(\mathcal{DR})$.

Proof. Let $(g, f_1, f_2), (G, F_1, F_2) \in \mathcal{DR}$. Then,

$$\begin{aligned}
\sigma(g, f_1, f_2) * \sigma(G, F_1, F_2) &= (g, f_2, f_1) * (G, F_2, F_1) \\
&= \left(gG(h), \frac{f_2}{h} F_2(h), \frac{f_1}{h} F_1(h) \right), \quad h = \sqrt{f_1 f_2} \\
&= \sigma \left(gG(h), \frac{f_1}{h} F_1(h), \frac{f_2}{h} F_2(h) \right) \\
&= \sigma((g, f_1, f_2) * (G, F_1, F_2)).
\end{aligned}$$

It is clear that σ is also one-to-one and onto. □

Although the automorphism group of the Riordan (or Double Riordan) group is not yet determined, the map defined in Theorem 12 appears to be generalizable to the k-Riordan group. Theorem 12 may suggest that a subset of the automorphism group $\mathrm{Aut}(k\mathcal{R})$ can be defined by automorphisms σ_k permuting the multiplier functions of an arbitrary array $(g, f_1, \ldots, f_k) \in k\mathcal{R}$. Furthermore, this subset would be isomorphic to the symmetric group, S_k.

4. Conclusion and future research

This paper answers the question posed by Davenport, Shapiro, and Woodson about finding an isomorphism between the Riordan group and a subgroup of the Double Riordan group, and we use the monomorphisms to extend known results for the Riordan group to the Double Riordan group.

We are interested in determining if there exists a monomorphism $\Delta : \mathcal{R} \to \mathcal{DR}$ that will map pseudo-involutions of the Riordan group to those in the Double Riordan group, in particular to $(1, -z, -z)$. If this map exists, then we would be able to generalize it to map Riordan arrays to pseudo-involutions of the k-Riordan group. Lastly, we are interested in exploring $\mathrm{Aut}(\mathcal{R})$, using those findings to understand $\mathrm{Aut}(k\mathcal{R})$ and extend results from this paper.

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