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Enumerative Combinatorics and Applications

ECA **6:4** (2026) Article #S2R30
https://doi.org/10.54550/ECA2026V6S4R30

A Determinant Parlor Trick

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Received: February 6, 2024, Accepted: December 19, 2025, Published: July 17, 2026

Abstract: Suppose that A and B are lower-triangular matrices and that A and B are inverses of each other. Let $\overline{A}$ be the Hessenberg matrix obtained by deleting the top row and the rightmost column of A. Then Cramer's rule tells us that

$$\det \overline{A} = \pm\big(\text{the entry at the bottom of the leftmost column of } B\big).$$

This fits neatly in the Riordan group, where we have lower-triangular matrices and group inverses. We illustrate this with examples. Some involve ordinary generating functions, and some involve exponential generating functions. In particular, this works neatly with pseudo-involutions. Some famous names associated with the examples include Pascal, Catalan, Bernoulli, and Stirling.

Keywords: Cramer's rule; Hessenberg determinant; Pseudo-involution; Riordan group
2020 Mathematics Subject Classification: 05A05; 05A10; 05A15

First, we will define a few of the items we will be using. The Riordan group is a group of infinite lower-triangular matrices defined by two generating functions, g and f:

$$f = f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots ,$$

$$g = g(z) = g_0 + g_1 z + g_2 z^2 + \cdots .$$

Also $g_0 \neq 0$ and $f_1 \neq 0$ so g is invertible and f has a compositional inverse. This compositional inverse is denoted $\overline{f}$. Let $R := (g, f)$ be the matrix determined by g and f, where

$$r_{n,k} = [z^n] g f^k,$$

with $k = 0, 1, 2, \ldots$. The identity is $(1, z)$, the usual matrix identity, and

$$(g, f)^{-1} = \left(\frac{1}{g(\overline{f})}, \overline{f}\right).$$

The Bell subgroup of the Riordan group is the set of all (g, zg). For more information about the Riordan group, see [2, 5, 11]. Some of our examples involve the Catalan numbers. We denote their generating function by

$$C = C(z) = 1 + zC^2 = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots .$$

Example 1. *Our first example will illustrate the ideas involved. We choose the Riordan group element (C^2, zC^2). Here $f = zC^2$, and an easy computation shows that*

$$\overline{f} = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + \cdots .$$

We are in the Bell subgroup, so

$$(C^2, zC^2)^{-1} = \left(\frac{1}{(1+z)^2}, \frac{z}{(1+z)^2}\right).$$

Writing out the first five rows of $(C^2, zC^2)^{-1}(C^2, zC^2)$ gives us

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ -4 & 10 & -6 & 1 & 0 \\ 5 & -20 & 21 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 \\ 42 & 48 & 27 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Trim this down to the leftmost column, and we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ -4 & 10 & -6 & 1 & 0 \\ 5 & -20 & 21 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now recall Cramer's rule, which tells us that

$$\frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ -4 & 10 & -6 & 1 & 0 \\ 5 & -20 & 21 & -8 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ -4 & 10 & -6 & 1 & 0 \\ 5 & -20 & 21 & -8 & 1 \end{vmatrix}} = 42 = \frac{\begin{vmatrix} -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ -4 & 10 & -6 & 1 \\ 5 & -20 & 21 & -8 \end{vmatrix}}{1}.$$

We have just computed the fifth term of a sequence of Hessenberg matrix determinants that starts

$$\begin{vmatrix} -2 \end{vmatrix} = -2, \qquad \begin{vmatrix} -2 & 1 \\ 3 & -4 \end{vmatrix} = 5, \qquad \begin{vmatrix} -2 & 1 & 0 \\ 3 & -4 & 1 \\ -4 & 10 & -6 \end{vmatrix} = -14, \ldots.$$

Thus, the leading principal minors of the infinite Hessenberg matrix that starts

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \\ 3 & -4 & 1 & 0 & 0 & \\ -4 & 10 & -6 & 1 & 0 & \cdots \\ 5 & -20 & 21 & -8 & 1 & \\ -6 & 35 & -56 & 36 & -10 & \\ & & \vdots & & & \end{bmatrix}$$

are

$$-2, 5, -14, 42, \ldots, (-1)^n C_{n+1}, \ldots.$$

Note that the elements of this proof consist of Cramer's rule and the fact that a lower-triangular matrix with all ones along the main diagonal has determinant one.

Example 2. *A slightly cleaner version eliminates all the minus signs. Using $(C^2, zC^2)(C^2, zC^2)^{-1}$ gives the dual result that the leading principal minors of*

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \\ 5 & 4 & 1 & 0 & 0 & \\ 14 & 14 & 6 & 1 & 0 & \cdots \\ 42 & 48 & 27 & 8 & 1 & \\ 132 & 165 & 110 & 44 & 10 & \\ & & \vdots & & & \end{bmatrix}$$

are $2, 3, 4, 5, \ldots$.

If we restore the top row and rotate the above matrix by 90°*, then we count Motzkin paths in which the level steps have weight* 2.

In general, we have the following proposition.

Proposition 3. *Let A and B be lower-triangular matrices with $A^{-1} = B$ and $a_{0,0} = 1$. Let $\overline{A}$ be the Hessenberg matrix obtained by removing the top row of A. Then the leading principal minors of $\overline{A}$ are the entries in the leftmost column of B, with alternating signs.*

Example 4. *The next example comes from Problem 3784 of the* American Mathematical Monthly *in 1933. It asks us to show that the determinant of*

$$\begin{bmatrix} \frac{1}{2!} & 1 & 0 & 0 & 0 & & \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & & \cdots \\ \vdots & \vdots & & \ddots & 1 & & \\ \frac{1}{(2k+1)!} & \frac{1}{(2k)!} & \frac{1}{(2k-1)!} & \ddots & \frac{1}{2!} & 1 & \\ \frac{1}{(2k+2)!} & \frac{1}{(2k+1)!} & \frac{1}{(2k)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & \end{bmatrix}$$

is 0 *for all positive integers k. We can learn more by looking at the Riordan group element*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2!} & 1 & 0 & 0 & 0 & 0 & \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & 0 & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & & \ddots & 1 & 0 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \ddots & \frac{1}{2!} & 1 & \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & \\ & & \vdots & & & & \end{bmatrix} = \left(\frac{e^z-1}{z}, z\right).$$

The inverse matrix is the Riordan array

$$\left(\frac{z}{e^z-1}, z\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & \\ \frac{1}{12} & -\frac{1}{2} & 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{12} & -\frac{1}{2} & 1 & 0 & \\ -\frac{1}{720} & 0 & \frac{1}{12} & -\frac{1}{2} & 1 & \\ 0 & -\frac{1}{720} & 0 & \frac{1}{12} & -\frac{1}{2} & \\ & & \vdots & & & \end{bmatrix} = \begin{bmatrix} B_0 & 0 & 0 & 0 & 0 & \\ B_1 & B_0 & 0 & 0 & 0 & \\ \frac{B_2}{2!} & B_1 & B_0 & 0 & 0 & \cdots \\ \frac{B_3}{3!} & \frac{B_2}{2!} & B_1 & B_0 & 0 & \\ \frac{B_4}{4!} & \frac{B_3}{3!} & \frac{B_2}{2!} & B_1 & B_0 & \\ \frac{B_5}{5!} & \frac{B_4}{4!} & \frac{B_3}{3!} & \frac{B_2}{2!} & B_1 & \\ & & \vdots & & & \end{bmatrix}.$$

Thus, our determinants from the Hessenberg matrix

$$\begin{bmatrix} \frac{1}{2!} & 1 & 0 & 0 & 0 & & \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & & \cdots \\ \vdots & \vdots & & \ddots & 1 & & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \ddots & \frac{1}{2!} & 1 & \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & \\ & & \vdots & & & & \end{bmatrix}$$

are $B_n/n!$, where the B_n are the Bernoulli numbers. The 1933 version uses the fact that

$$0 = B_3 = B_5 = B_7 = \cdots.$$

Example 5. *Here is a third example. We look at a Fibonacci matrix and its inverse to obtain a well-known result and another less well-known result:*

$$\left(\frac{1}{1-z-z^2}, z\right) = F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 1 & 0 & \\ 5 & 3 & 2 & 1 & 1 & \\ 8 & 5 & 3 & 2 & 1 & \\ & \vdots & & & & \end{bmatrix}, \qquad F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & \\ -1 & -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & -1 & 1 & 0 & \\ 0 & 0 & -1 & -1 & 1 & \\ 0 & 0 & 0 & -1 & -1 & \\ & \vdots & & & & \end{bmatrix}.$$

The well-known result [13] is that

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \\ -1 & -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & -1 & 1 & 0 & \\ 0 & 0 & -1 & -1 & 1 & \\ 0 & 0 & 0 & -1 & -1 & \\ & \vdots & & & & \end{bmatrix}$$

has leading principal minors

$$-1, 2, -3, 5, -8, \dots, (-1)^n F_{n+1}, \dots.$$

Perhaps less well known is that the leading principal minors of

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 1 & 0 & \\ 5 & 3 & 2 & 1 & 1 & \\ 8 & 5 & 3 & 2 & 1 & \\ & \vdots & & & & \end{bmatrix}$$

are $1, -1, 0, 0, 0, \dots$. *This follows from*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 5 & 3 & 2 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix}.$$

Both results are easily proved directly.

Example 6. *Here are a pair of classical examples. We know that the Stirling numbers of the second kind and the signed Stirling numbers of the first kind are inverses of each other. For instance, again using the first few rows to illustrate the ideas, we have*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In Riordan group terminology, we are dealing with exponential generating functions, and we have

$$[e^z, e^z - 1]\left[\frac{1}{1+z}, \ln(1+z)\right] = [1, z].$$

Again, see [2, 5, 11] for more information regarding exponential Riordan arrays. Thus,

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \\ 1 & 15 & 25 & 10 \end{bmatrix} = 24,$$

and the determinant sequence is

$$1, 2, 6, 24, \dots, n!, \dots.$$

Similarly,

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \\ 24 & 50 & 35 & 10 \end{bmatrix} = 1,$$

with $1, 1, 1, 1, \dots$ *as our determinant sequence.*

Example 7. *Here is a special case that is entertaining. Let X be a pseudo-involution, so that X^{-1} is the same as X with a sign change on alternate diagonals; for more information, see [3, 7]. Then the determinant is the bottom-left entry of the Hessenberg matrix. Again, an example may be the best way to see what is happening. The best-known pseudo-involution is probably Pascal's matrix, and, for instance,*

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{vmatrix} = 1.$$

If we eliminate the first two rows and columns, we still have a pseudo-involution, and

$$\begin{vmatrix} 3 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 10 & 10 & 5 & 1 \\ 15 & 20 & 15 & 6 \end{vmatrix} = 15.$$

Example 8. *We conclude with a result attributed to Wronski [12] in 1811. This is a long treatise mixing a little mathematics with a lot of political philosophy. For more modern references in English, see Riordan [8] and Henrici [6]. Suppose we have a function*

$$A(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

and we want to find the coefficients of $B(z) = 1/A(z)$. We form the Riordan group pair $(A(z), z)$ and its inverse $(B(z), z)$. We can express the coefficients b_n in terms of the a_i. In fact, $1, -b_1, b_2, -b_3, \ldots$ is the leading-principal-minor sequence of the Hessenberg matrix obtained from $(A(z), z)$, and

$$b_n = (-1)^n \begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & \\ a_2 & a_1 & 1 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 1 & 0 & \\ a_4 & a_3 & a_2 & a_1 & 1 & \\ & \ddots & \ddots & \ddots & & \\ a_n & a_{n-1} & \cdots & & a_1 & \end{vmatrix}.$$

Again, our pair of Riordan arrays gives us another well-known result. We introduce a variable x and look at the pair

$$A = (1 - 2xz + z^2, z) \qquad \text{and} \qquad B = \left(\frac{1}{1 - 2xz + z^2}, z\right).$$

This tells us that

$$U_n(x) = (-1)^n \det \begin{bmatrix} -2x & 1 & 0 & 0 & & \\ 1 & -2x & 1 & 0 & & \\ 0 & 1 & -2x & 1 & & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ & & & -2x & 1 & \\ & & & 1 & -2x & \end{bmatrix}_{n \times n}.$$

Thus, up to alternating signs, the principal minors are the Chebyshev polynomials of the second kind, whose generating function is

$$\frac{1}{1 - 2xz + z^2} = \sum_{n \geq 0} U_n(x) z^n.$$

See also Andreescu and Gelca [1] and Comtet [4].

In summary, we note two things. We can easily evaluate a large class of Hessenberg determinants, and we have a nice application of Riordan group techniques.

Acknowledgment: I would like to thank Drew Dickenson and the referees for their careful reading and helpful comments.

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