ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **6:4** (2026) Article #S2R29
https://doi.org/10.54550/ECA2026V6S4R29

Hills and Hitting Times

Drew Dickenson

Department of Mathematics, Howard University
Email: drew.dickenson@howard.edu

Received: November 3, 2025, Accepted: December 19, 2025, Published: July 17, 2026

Abstract: We study the relationship between leaves in ordered trees and first returns of lattice paths. We introduce uniform-updegree requirement (UUR) trees and their associated generating functions, and give combinatorial explanations of the relations among these functions. We use generating functions to prove that, for every $n \geq 1$, the total first return length of the corresponding Łukasiewicz paths of length n equals the total number of leaves in UUR trees with n edges. As a special case, we compute the first return statistic for weighted Motzkin paths. We then consider a more general Motzkin model in which horizontal steps on the x-axis have weight h_0, horizontal steps above the axis have weight h, and down steps have weight d. We transform weighted Schröder paths into weighted Motzkin paths and apply the resulting formulas to Schröder-type numbers. Finally, we compute the associated Hankel determinants and use the formula to solve Problem A6 of the 2024 William Lowell Putnam Mathematical Competition.

Keywords: Hankel determinants; Leaves; Lukasiewicz paths; Motzkin paths; Ordered trees; Schröder paths; Riordan arrays
2020 Mathematics Subject Classification: Primary 05A15; Secondary 05A19; 05C05

1. Introduction

In this paper, we study the relationship among three objects: leaves in ordered trees, first returns of lattice paths, and the hitting time subgroup of the Riordan group [4]. We begin with an infinite sequence

$$a_0, a_1, a_2, \dots$$

of nonnegative integers and its generating function

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

We draw each rooted ordered tree with its root at the bottom. The *updegree* of a vertex is its number of children, equivalently, the number of edges directed away from the root. Thus, a_k is the number of available colors for a vertex of updegree k. In the most common uncolored setting, $a_k = 1$ means that updegree k is allowed, whereas $a_k = 0$ means that it is forbidden.

Contributions and organization. Section 2 develops the UUR generating functions and the associated Riordan arrays arising from the weight sequence $A(z)$. Section 3 proves the main equidistribution: for each $n \geq 1$, the total number of leaves among UUR trees with n edges equals the total first return length among the corresponding Łukasiewicz paths of length n. Section 4 treats a nonhomogeneous Motzkin model with distinct weights for horizontal steps on and above the x-axis. Section 5 transfers weighted Schröder models to this setting by contracting their continued fractions. Section 6 records the resulting Hankel determinant evaluation, a classical formula discussed, for example, by Aigner [1] and Kim and Stanton [10], and applies it to give a Motzkin/Hankel proof of Putnam 2024 A6. Throughout, we cite standard results and include proofs when they clarify the combinatorial viewpoint.

Here are three basic examples.

A. The best-known and most studied example is the class of ordered trees, where $a_k = 1$ for every k. The number of such trees with n edges is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

See Stanley [12] for many combinatorial interpretations of the Catalan numbers.

B. If $a_0 = a_1 = a_2 = 1$ and $a_k = 0$ for $k \geq 3$, we obtain Motzkin trees, in which every vertex has updegree 0, 1, or 2.

C. Incomplete binary trees, which arise naturally in data structures, are defined by $a_0 = a_2 = 1$ and $a_1 = 2$, because a unary vertex may have either a left child or a right child.

We illustrate several of the ideas using the second example. Here

$$A(z) = 1 + z + z^2.$$

Let f satisfy

$$f = zA(f) = z(1 + f + f^2),$$

and define m by $f = zm$. Then m is the Motzkin number generating function:

$$m = 1 + zm + z^2m^2 = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2} = 1 + z + 2z^2 + 4z^3 + 9z^4 + \cdots .$$

The total number of leaves among Motzkin trees is counted by

$$E(z) = \frac{1}{\sqrt{1 - 2z - 3z^2}} = 1 + z + 3z^2 + 7z^3 + 19z^4 + \cdots .$$

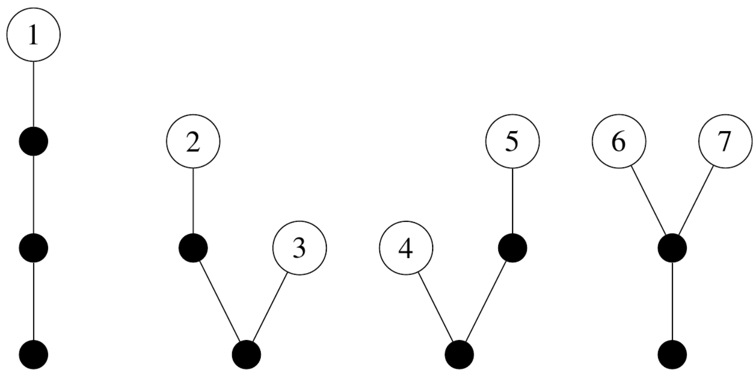

Figure 1: The seven leaves among the Motzkin trees with three edges.

A Motzkin path is a lattice path from $(0, 0)$ to $(n, 0)$ using up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down steps $D = (1, -1)$, never passing below the x-axis. The hitting time statistic has a simple queueing interpretation: in each unit of time, exactly one of three events occurs. An event U means that one customer enters a store, an event D means that one customer leaves, and an event H means that no customer enters or leaves. The hitting time is the first positive time at which the store is empty. In a continuous-time birth–death analogue, exponential waiting times lead to formulas involving Bessel functions; see Feller [6] and Grimmett and Stirzaker [8].

Definition 1. The *hitting time*, or *first return time*, of a nonempty lattice path is the number of steps from the origin through its first return to the x-axis.

Let

$$g(z) = \sum_{n \geq 0} g_n z^n$$

be the generating function for all paths of length n that start and end on the x-axis and remain weakly above it. Let

$$F(z) = \sum_{n \geq 1} f_n z^n$$

be the generating function for *first return blocks*: nonempty paths whose first return to the axis occurs at their final step. The usual first return decomposition gives

$$g = 1 + Fg, \qquad F = \frac{g - 1}{g}.$$

The series $zF'(z)$ weights each first return block by its length. We therefore define

$$G(z) := zF'(z)g(z),$$

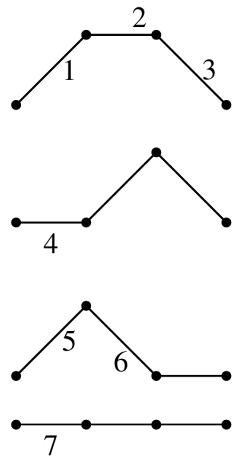

Figure 2: The seven edges involved in the first returns of the four Motzkin paths with three edges.

so that $[z^n]G(z)$ is the total first return time summed over all paths of length n. Differentiating $F = (g-1)/g$ gives

$$G(z) = zF'(z)g(z) = \frac{zg'(z)}{g(z)}.$$

For ordinary Motzkin paths, $g = m$ and

$$F = z + z^2 m.$$

Consequently,

$$G = zF'm = z + 3z^2 + 7z^3 + 19z^4 + 51z^5 + \cdots.$$

Thus, for every positive length, the total first return time agrees with the total number of leaves in the corresponding Motzkin trees; the leaf series has the additional constant term 1 from the one-vertex tree. Figures 1 and 2 illustrate the equality

$$1 + 2 + 2 + 2 = 3 + 2 + 1 + 1.$$

For $n = 4$, the corresponding equality is

$$1 + 2 + 2 + 2 + 2 + 3 + 2 + 2 + 3 = 19 = 4 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1.$$

2. UUR trees

A *uniform updegree requirement tree*, or UUR tree [9], is a rooted ordered tree in which every vertex uses the same set of allowed updegrees, with the same color multiplicities a_k.

For a fixed UUR class, let

$$T(z) = \sum_{n \geq 0} t_n z^n,$$

where t_n is the number of trees with n edges. Each edge contributes a factor of z. Let $V(z)$ count the same trees with a marked vertex, and let $L(z)$ count them with a marked leaf. Let $L_1(z)$ count trees with a leaf marked at height one and, in general, let $L_k(z)$ count trees with a leaf marked at height k.

2.1 Riordan arrays and notation

A Riordan array is a lower-triangular matrix $R = (r_{n,k})_{n,k \geq 0}$ determined by a pair of formal power series (g, f) with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$, via

$$r_{n,k} = [z^n]g(z)f(z)^k.$$

The Fundamental Theorem of Riordan Arrays states that R acts on a series $B(z)$ by

$$(g, f)B(z) = g(z)B(f(z)).$$

In particular, the k-th column of (g, f) has generating function $g(z)f(z)^k$. The Bell subgroup consists of arrays of the form (g, zg); in our setting, the basic Bell array is (T, zT).

We also use the following standard subgroup.

Definition 2. The *hitting time subgroup* of the Riordan group consists of pairs of the form

$$\left(\frac{zf'(z)}{f(z)}, f(z)\right).$$

Table 1 lists several familiar UUR classes and their A-sequences.

Type of tree	A-sequence	$A(z)$
Ordered trees	$1, 1, 1, 1, 1, 1, 1, \ldots$	$\frac{1}{1-z}$
Motzkin trees	$1, 1, 1, 0, 0, 0, 0, \ldots$	$1+z+z^2$
Incomplete binary trees	$1, 2, 1, 0, 0, 0, 0, \ldots$	$1+2z+z^2$
Complete binary trees	$1, 0, 1, 0, 0, 0, 0, \ldots$	$1+z^2$
Even trees	$1, 0, 1, 0, 1, 0, 1, \ldots$	$\frac{1}{1-z^2}$
Incomplete ternary trees	$1, 3, 3, 1, 0, 0, 0, \ldots$	$1+3z+3z^2+z^3$
Complete ternary trees	$1, 0, 0, 1, 0, 0, 0, \ldots$	$1+z^3$
Schröder trees	$1, 2, 2, 2, 2, 2, 2, \ldots$	$\frac{1+z}{1-z}$
Spoiled-child trees	$1, 2, 1, 1, 1, 1, 1, \ldots$	$\frac{1+z-z^2}{1-z}$
Hex trees	$1, 3, 1, 0, 0, 0, 0, \ldots$	$1+3z+z^2$
Gamma trees	$1, 0, 1, 1, 1, 1, 1, \ldots$	$\frac{1-z+z^2}{1-z}$
Simple trees (paths)	$1, 1, 0, 0, 0, 0, 0, \ldots$	$1+z$

Table 1: UUR trees and their A-sequences.

2.2 Fundamental identities for UUR trees

Theorem 1 (Functional equation for T)**.** *For any UUR tree class,*

$$T = A(zT).$$

Proof. A UUR tree consists of a root with k ordered subtrees, for some allowed k. Each root-to-subtree edge contributes z, and each subtree contributes $T(z)$. Thus, a root of updegree k contributes $a_k(zT)^k$. Summing over $k \geq 0$ gives

$$T(z) = \sum_{k\geq 0} a_k(zT(z))^k = A(zT(z)).$$

□

Theorem 2 (Vertex generating function)**.** *For any UUR tree class,*

$$V = (zT)'.$$

Proof. A tree with n edges has $n+1$ vertices. Hence,

$$V(z) = \sum_{n\geq 0}(n+1)t_n z^n = \frac{d}{dz}\left(\sum_{n\geq 0} t_n z^{n+1}\right) = (zT)'.$$

□

Theorem 3 (Vertex–leaf relation)**.** *For any UUR tree class,*

$$V = TL.$$

Proof. Given a tree with a marked vertex v, cut the tree at v. The part below v, including v, becomes a UUR tree with a marked leaf, counted by L, while the subtree rooted at v is an arbitrary UUR tree, counted by T. This decomposition is bijective. □

Theorem 4 (Leaves at height k)**.** *For every $k \geq 0$,*

$$L_k = L_1^k.$$

Proof. For $k = 0$, the only tree with a leaf marked at height zero is the one-vertex tree, so $L_0 = 1$. For $k \geq 1$, follow the unique path from the root to the marked leaf and cut immediately above each of its first $k-1$ vertices. This decomposes the tree uniquely into an ordered sequence of k UUR trees, each with a leaf marked at height one. Therefore, $L_k = L_1^k$. □

Theorem 5 (Total leaf generating function)**.** *For any UUR tree class,*

$$L = \frac{1}{1 - L_1}.$$

Proof. Summing over the height of the marked leaf and using Theorem 4 gives

$$L = \sum_{k \geq 0} L_k = \sum_{k \geq 0} L_1^k = \frac{1}{1 - L_1}.$$

□

Theorem 6 (Leaves at height one)**.** *For any UUR tree class,*

$$L_1 = zA'(zT) = \frac{zT'}{V}.$$

Proof. If the root has updegree k, there are k choices for the edge leading to the marked leaf, and the remaining $k - 1$ edges lead to arbitrary UUR subtrees. Hence,

$$L_1(z) = z \sum_{k \geq 0} k a_k (zT(z))^{k-1} = zA'(zT(z)).$$

For the second expression, Theorems 2 and 3 give

$$V = (zT)' = T + zT' = TL.$$

Thus, $zT' = T(L - 1)$ and

$$\frac{zT'}{V} = \frac{T(L-1)}{TL} = \frac{L-1}{L} = L_1,$$

where the final equality follows from $L = 1/(1 - L_1)$. □

Theorem 7 (Total leaf height)**.** *For any UUR tree class, the generating function for the total height of all leaves is*

$$L(L - 1).$$

Proof. The required generating function is

$$\sum_{k \geq 0} kL_k = \sum_{k \geq 0} kL_1^k = \frac{L_1}{(1 - L_1)^2}.$$

Since $L = 1/(1 - L_1)$, we have $L_1 = 1 - 1/L$, and therefore

$$\frac{L_1}{(1 - L_1)^2} = \frac{1 - 1/L}{(1/L)^2} = L(L - 1).$$

Equivalently, the Riordan array $(1, L_1)$ sends the sequence with generating function $z/(1 - z)^2$ to the total leaf height generating function. □

Corollary 1 (Statistics for ordered trees)**.** *For ordered trees, where $A(z) = 1/(1 - z)$, and for every $n \geq 1$:*

1. *the average number of leaves is $(n + 1)/2$, so half of the vertices are leaves;*
2. *the total leaf height over all ordered trees with n edges is 4^{n-1};*
3. *the average number of leaves at height one, equivalently hills in the corresponding Dyck paths, is 1;*
4. *the average root degree is $3n/(n + 2)$.*

Proof. Here $T = C$, the Catalan generating function, and

$$V = (zC)' = \frac{1}{\sqrt{1-4z}}, \qquad L = \frac{V}{C}.$$

For $n \geq 1$,

$$[z^n]L = \frac{1}{2}\binom{2n}{n} = \frac{n+1}{2}C_n,$$

which proves the first assertion. Moreover,

$$L(L-1) = \frac{z}{1-4z},$$

which proves the second. Since $L_1 = zA'(zC) = zC^2 = C - 1$, the coefficient of z^n in L_1 is C_n, proving the third. Finally, marking an edge incident with the root gives the generating function $zCA'(zC) = zC^3$; coefficient extraction yields an average root degree of $3n/(n+2)$. □

3. First returns

Definition 3. A Łukasiewicz path is a lattice path in $\mathbb{N}^2$ that starts at the origin, ends on the x-axis, never passes below it, and uses steps

$$\{(1, -k) : k \geq -1\}.$$

The step $(1, -k)$ has weight a_{k+1}.

The standard tree–path correspondence sends UUR trees with n edges to weighted paths of length n of the type defined above. Their common generating function is T. More generally, let g be the generating function for a class of such excursions, and let F count the nonempty excursions that end at their first return to the x-axis. Then

$$g = 1 + Fg, \qquad F = \frac{g-1}{g}, \qquad F' = \frac{g'}{g^2}.$$

Define G to be the generating function for the total first return time over all excursions. Since zF' marks the length of a first return block and the remaining suffix is arbitrary,

$$G = zF'g = \frac{zg'}{g}.$$

In the UUR setting, $g = T$, so

$$G = \frac{zT'}{T} = L - 1.$$

Therefore, for every $n \geq 1$, the total first return time among the corresponding Łukasiewicz paths of length n equals the total number of leaves among UUR trees with n edges. The constant term differs because the one-vertex tree has one leaf, while the empty path has no positive first return. The pair

$$\left(\frac{zf'}{f}, f\right)$$

is precisely the general form of an element of the hitting time subgroup.

3.1 Generalized Motzkin paths

We now consider (h, d)-Motzkin paths, in which every horizontal step has weight h, every down step has weight d, and every up step has weight 1. Let J be their generating function. The first-step decomposition gives

$$J = 1 + hzJ + dz^2J^2 = \frac{1 - hz - \sqrt{\Delta(z)}}{2dz^2},$$

where

$$\Delta(z) := (1 - hz)^2 - 4dz^2 = 1 - 2hz + (h^2 - 4d)z^2.$$

The corresponding UUR trees have

$$A(z) = 1 + hz + dz^2, \qquad A'(z) = h + 2dz.$$

Thus,

$$L_1 = zA'(zJ) = hz + 2dz^2J = 1 - \sqrt{\Delta(z)},$$

and

$$L = \frac{1}{1-L_1} = \frac{1}{\sqrt{\Delta(z)}}.$$

Equivalently,

$$J = L(1 - dz^2J^2).$$

3.2 Example: ordinary Motzkin paths

For $h = d = 1$, we have $J = m$ and

$$F = \frac{m-1}{m}, \qquad F' = \frac{m'}{m^2}, \qquad G = \frac{zm'}{m}.$$

Moreover,

$$G = L - 1 = \frac{1}{\sqrt{1-2z-3z^2}} - 1 = z + 3z^2 + 7z^3 + 19z^4 + \cdots .$$

4. Motzkinable sequences

Definition 4. A sequence is *Motzkinable* if its generating function counts weighted Motzkin paths with parameters (h_0, h, d), where up steps have weight 1, horizontal steps on the x-axis have weight h_0, horizontal steps above the x-axis have weight h, and down steps have weight d.

Let $J = J_{h,d}$ be the homogeneous elevated excursion generating function from Section 3.1. The generating function $M = M_{h_0,h,d}$ for the nonhomogeneous model satisfies

$$M = 1 + h_0zM + dz^2JM,$$

so

$$M = \frac{1}{1 - h_0z - dz^2J} = \frac{2}{1 + (h-2h_0)z + \sqrt{\Delta(z)}}.$$

Its first return block series and total first return time series are, respectively,

$$F_{h_0,h,d} = h_0z + dz^2J, \qquad G_{h_0,h,d} = zF'_{h_0,h,d}M.$$

When $h_0 = h$, the model is homogeneous, $M = J$, and the identity $G = L - 1$ from Section 3 applies.

Several classical sequences are Motzkinable:

- Motzkin: $(h_0, h, d) = (1, 1, 1)$;
- Dyck (aerated Catalan): $(0, 0, 1)$;
- Catalan: $(1, 2, 1)$;
- Fine: $(0, 2, 1)$;
- small Schröder: $(1, 3, 2)$;
- large Schröder: $(2, 3, 2)$.

5. The small Schröder numbers in many guises

5.1 Classical definitions and generating functions

The large Schröder numbers r_n count paths from $(0, 0)$ to $(2n, 0)$ using up steps $U = (1, 1)$, horizontal steps $H = (2, 0)$, and down steps $D = (1, -1)$, never passing below the x-axis. Their generating function satisfies

$$r = 1 + zr + zr^2 = \frac{1 - z - \sqrt{1-6z+z^2}}{2z}.$$

The small Schröder numbers s_n forbid horizontal steps on the x-axis. Their generating function is

$$s = \frac{r+1}{2} = 1 + zrs = \frac{1 + z - \sqrt{1-6z+z^2}}{4z}.$$

These numbers occur in many combinatorial settings, including:

- dissections of a convex $(n+2)$-gon by nonintersecting diagonals, including the trivial dissection;
- ways to insert parentheses in a string of $n+1$ symbols when operations of arbitrary arity at least two are allowed;
- increasing tableaux of shape (n,n), that is, fillings whose rows and columns are strictly increasing and whose entries form an initial segment of the positive integers.

5.2 From Schröder paths to Motzkin paths

Consider weighted Schröder paths in which up steps have weight 1, horizontal steps on the x-axis have weight H_0, horizontal steps above the axis have weight H, and down steps have weight D. Their generating function has the Stieltjes continued fraction

$$\mathcal{S} = \cfrac{1}{1 - H_0 z - \cfrac{Dz}{1 - Hz - \cfrac{Dz}{1 - Hz - \ddots}}}.$$

Contracting adjacent levels of this continued fraction gives the Jacobi continued fraction

$$\mathcal{S} = \cfrac{1}{1 - (H_0+D)z - \cfrac{(H+D)Dz^2}{1 - (H+2D)z - \cfrac{(H+D)Dz^2}{1 - (H+2D)z - \ddots}}}.$$

This is the generating function of a Motzkin model with parameters

$$h_0 = H_0 + D, \qquad h = H + 2D, \qquad d = (H+D)D.$$

Combinatorially, the contraction groups the two successive height changes in a Schröder excursion into a single Motzkin transition; the three possible local outcomes become a level step on the axis, a level step above it, or an up–down pair. Related transformations appear in [2, 3, 5, 15].

Theorem 8 (Motzkin representation of small Schröder numbers)**.** *The small Schröder numbers form a Motzkinable sequence with parameters*

$$(h_0, h, d) = (1, 3, 2).$$

Thus, up steps have weight 1*, horizontal steps on the x-axis have weight* 1*, horizontal steps above the axis have weight* 3*, and down steps have weight* 2*.*

Proof. For small Schröder paths, $(H_0, H, D) = (0, 1, 1)$. The contraction above gives

$$(h_0, h, d) = (0+1, 1+2, (1+1)\cdot 1) = (1, 3, 2).$$

Equivalently, if J counts elevated Motzkin excursions, then

$$J = 1 + 3zJ + 2z^2J^2 = \frac{1 - 3z - \sqrt{1 - 6z + z^2}}{4z^2},$$

while the boundary generating function s satisfies

$$s = 1 + zs + 2z^2Js.$$

Substitution yields

$$s = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z}.$$

□

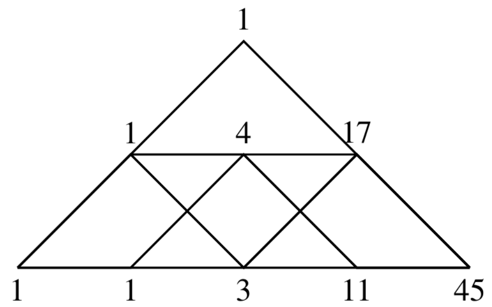

Figure 3: The 45 small Schröder paths of length 4, represented as Motzkin paths with weights $(1, 3, 2)$. The numbers indicate cumulative path counts.

5.3 UUR tree interpretation

The small Schröder numbers also admit a UUR tree interpretation. For the Bell array (s, zs), the A-sequence is

$$1, 1, 2, 4, 8, 16, \ldots,$$

with generating function

$$A(z) = \frac{1-z}{1-2z}.$$

Theorem 9 (UUR trees for small Schröder numbers)**.** *The small Schröder numbers count UUR trees in which the leftmost edge from every nonleaf vertex is black and every other edge from that vertex may be red or black.*

Proof. The tree equation $s = A(zs)$ becomes

$$s = \frac{1-z}{1-2z} \circ (zs) = \frac{1-zs}{1-2zs}.$$

Rearranging gives

$$s = 1 + zs(2s-1) = 1 + zrs,$$

because $r = 2s - 1$. This is the defining relation for the small Schröder numbers. □

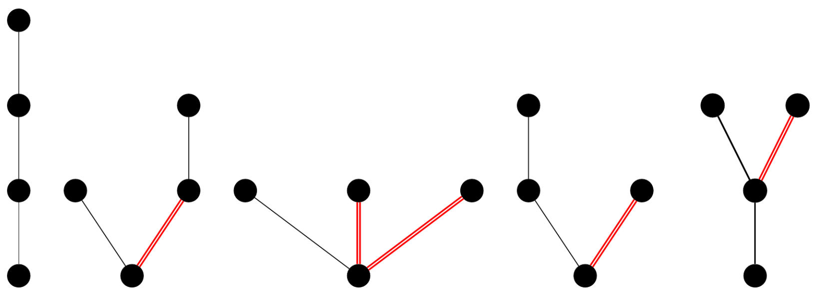

Figure 4: The eleven trees with three edges, corresponding to $s_3 = 1 + 2 + 4 + 2 + 2 = 11$.

5.4 Hankel determinants

For a sequence $(m_n)_{n\geq 0}$, define its Hankel determinants by

$$H_n := \det(m_{i+j})_{0\leq i,j\leq n-1}.$$

The following classical evaluation follows from the Lindström–Gessel–Viennot method; see also Aigner [1] and Kim and Stanton [10].

Theorem 10 (Hankel determinants for Motzkinable sequences)**.** *Let $(m_n)_{n\geq 0}$ be a Motzkinable sequence with parameters (h_0, h, d). Then*

$$H_n = d^{\binom{n}{2}}.$$

Proof. Consider the standard weighted Motzkin-path network in which up steps have weight 1 and down steps have weight d. Let the starting points be $A_i = (-i, 0)$ and the ending points be $B_j = (j, 0)$ for $0 \le i, j \le n-1$. The total weight of paths from A_i to B_j is m_{i+j}.

By the Lindström–Gessel–Viennot lemma [7], the determinant is the signed weight sum of nonintersecting path families. The only such family matches A_i with B_i. To avoid the inner paths, the path from A_i to B_i must consist of i nested up steps followed by i nested down steps. Its weight is d^i. Therefore, the unique family has total weight

$$\prod_{i=0}^{n-1} d^i = d^{0+1+\cdots+(n-1)} = d^{\binom{n}{2}},$$

which proves the result. □

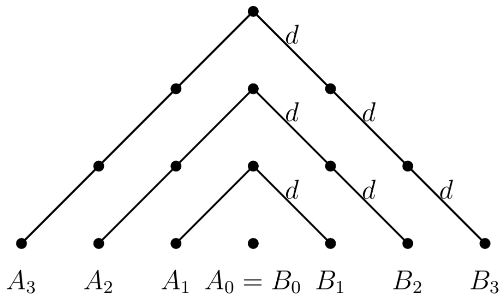

Figure 5: The unique nonintersecting path family in the Hankel determinant evaluation.

For the small Schröder numbers, $d = 2$, so

$$\det(s_{i+j})_{0 \le i,j \le n-1} = 2^{\binom{n}{2}}.$$

6. Application: Putnam 2024 A6

We now apply the Motzkinable-sequence framework to Problem A6 of the 2024 William Lowell Putnam Mathematical Competition [14].

6.1 Problem statement

Let $c_0, c_1, c_2, \ldots$ be defined by

$$\frac{1 - 3x - \sqrt{1 - 14x + 9x^2}}{4} = \sum_{k \ge 0} c_k x^k$$

for sufficiently small x. For a positive integer n, let

$$A = (c_{i+j-1})_{1 \le i,j \le n}.$$

Find $\det A$.

6.2 Solution via Motzkinable sequences

Theorem 11 (Putnam A6 solution)**.** *The determinant of* $A = (c_{i+j-1})_{1 \le i,j \le n}$ *is*

$$\det A = 10^{\binom{n}{2}}.$$

Proof. The constant term is $c_0 = 0$. Define

$$g(x) := \frac{1}{x} \sum_{k \ge 0} c_k x^k = \frac{1 - 3x - \sqrt{(1-3x)^2 - 8x}}{4x}.$$

Then

$$2xg^2 - (1 - 3x)g + 1 = 0,$$

or, equivalently,

$$g = 1 + 3xg + 2xg^2.$$

Thus, g is the generating function for weighted Schröder paths with

$$(H_0, H, D) = (3, 3, 2).$$

By the contraction in Section 5.2, the same sequence is Motzkinable with

$$(h_0, h, d) = (H_0 + D, H + 2D, (H + D)D) = (5, 7, 10).$$

If $m_k = [x^k]g(x)$, then $m_k = c_{k+1}$, and hence

$$A = (c_{i+j-1})_{1\le i,j\le n} = (m_{i+j-2})_{1\le i,j\le n}.$$

Theorem 10 therefore gives

$$\det(c_{i+j-1})_{1\le i,j\le n} = 10^{\binom{n}{2}}.$$

□

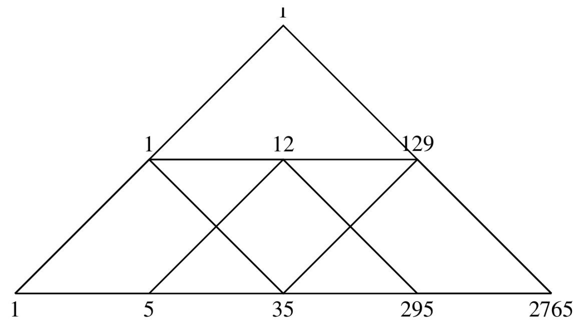

Figure 6: Motzkin paths with parameters $(5, 7, 10)$ through length 4. The bottom row begins with $c_1, c_2, c_3, c_4, c_5 = 1, 5, 35, 295, 2765$.

6.3 Remark

This solution illustrates the usefulness of the Motzkinable-sequence framework: once a sequence is represented by parameters (h_0, h, d), its Hankel determinants are immediately given by $d^{\binom{n}{2}}$.

7. Summary

We have developed a unified framework relating UUR trees, first return statistics of Łukasiewicz-type and Motzkin-type paths, and Riordan arrays arising from a general weight sequence $A(z)$. For every $n \ge 1$, the total number of leaves among UUR trees with n edges equals the total first return length among the corresponding paths of length n. We also treated Motzkin paths with distinct horizontal weights on and above the axis, explained the continued-fraction contraction that transfers weighted Schröder paths to this model, and obtained the classical Hankel determinant formula

$$H_n = d^{\binom{n}{2}}.$$

This interpretation yields a transparent Motzkin/Hankel solution of Putnam 2024 A6.

Acknowledgments: The author thanks Louis Shapiro for his helpful advice and the referees for their careful reading and helpful comments.

References

[1] M. Aigner, *A Course in Enumeration*, Graduate Texts in Mathematics, Vol. 238, Springer, Berlin, 2007.

[2] W. Y. C. Chen, S. H. F. Yan, and L. L. M. Yang, *Identities from weighted Motzkin paths*, Adv. in Appl. Math. 41 (2008), 329–334.

[3] Z. Chen and H. Pan, *Identities involving weighted Catalan, Schröder, and Motzkin paths*, Adv. in Appl. Math. 86 (2017), 81–98.

[4] G.-S. Cheon, H. Kim, and L. W. Shapiro, *The hitting time subgroup, Łukasiewicz paths and Faber polynomials*, European J. Combin. 32 (2011), 82–91.

[5] E. Deutsch, *Dyck path enumeration*, Discrete Math. 204 (1999), 167–202.

[6] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York, 1971.

[7] I. Gessel and X. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. 58 (1985), 300–321.

[8] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, Oxford University Press, Oxford, 2001.

[9] H. Kim and L. W. Shapiro, *Pick two points in a tree*, J. Korean Math. Soc. 56 (2019), 1247–1263.

[10] J. S. Kim and D. Stanton, *Combinatorics of orthogonal polynomials of type R_I*, Ramanujan J. 61 (2023), 329–390.

[11] L. W. Shapiro, S. Getu, W.-J. Woan, and L. C. Woodson, *The Riordan group*, Discrete Appl. Math. 34 (1991), 229–239.

[12] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, Cambridge, 1999.

[13] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

[14] William Lowell Putnam Mathematical Competition, The 85th William Lowell Putnam Mathematical Competition (2024): Problems and solutions, competition archive, 2024.

[15] S. H. F. Yan, *From* $(2,3)$*-Motzkin paths to Schröder paths*, J. Integer Seq. 10 (2007), Article 07.9.1.